A NOTE ON L-PACKETS AND ABELIAN VARIETIES OVER LOCAL FIELDS

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Abstract. A polarized abelian variety (X, λ) of dimension g and good reduction over a local field K determines an admissible representation of $\mathrm{GSpin}_{2g+1}(K)$. We show that the restriction of this representation to $\mathrm{Spin}_{2g+1}(K)$ is reducible if and only if X is isogenous to its twist by the quadratic unramified extension of K. When $g = 1$ and $K = \mathbb{Q}_p$, we recover the well-known fact that the admissible $\mathrm{GL}_2(K)$ representation attached to an elliptic curve E with good reduction is reducible upon restriction to $\mathrm{SL}_2(K)$ if and only if E has supersingular reduction.

Introduction

Consider an elliptic curve $E/\mathbb{Q}_p$ with good reduction. Let π_E be the unramified principal series representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with the same Euler factor as E. Although π_E is irreducible, the restriction of π_E from $\mathrm{GL}_2(\mathbb{Q}_p)$ to its derived group, $\mathrm{SL}_2(\mathbb{Q}_p)$, need not be irreducible. In fact, it is not hard to show that $\pi_E|_{\mathrm{SL}_2(\mathbb{Q}_p)}$ is reducible if and only if the reduction of E is supersinglar, see [1, 2.1] for example.

This note generalizes the observation above, as follows. Let K be a non-Archimedean local field with finite residue field and let (X, λ) be a polarized abelian variety over K of dimension g with good reduction. Fix a rational prime ℓ invertible in the residue field of K. Then the associated Galois representation on the ℓ-adic Tate module of X takes values in $\mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda) \cong \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$. The eigenvalues of the image of Frobenius under this unramified representation determine an irreducible principal series representation $\pi_{X,\lambda}$ of $\mathrm{GSpin}_{2g+1}(K)$ with the same Euler factor as X. Note that the dual group to GSpin_{2g+1} is GSp_{2g}; note also that $\mathrm{GSpin}_3 \cong \mathrm{GL}_2$ and $\mathrm{GSpin}_5 \cong \mathrm{GSp}_4$, accidentally. We show that the restriction of $\pi_{X,\lambda}$ from $\mathrm{GSpin}_{2g+1}(K)$ to its derived group $\mathrm{Spin}_{2g+1}(K)$ is reducible if and only if X is isogenous to its twist by the quadratic unramified extension of K.

In this note we also identify the Langlands parameter $\phi_{X,\lambda}$ for $\pi_{X,\lambda}$ and then show that the corresponding L-packet $\Pi_{X,\lambda}$ contains the equivalence class of $\pi_{X,\lambda}$ only. Then we show that we can detect when X is isogenous to its quadratic unramified twist directly from the local L-packet $\Pi^{\mathrm{der}}_{X,\lambda}$ determined by transferring the Langlands parameter $\phi_{X,\lambda}$ to the derived group $\mathrm{Spin}_{2g+1}(K)$ of $\mathrm{GSpin}_{2g+1}(K)$.

It is natural to ask how the story above extends to include abelian varieties X over local fields which do not have good reduction, keeping track of the relation between the ℓ-adic Tate module $T_\ell X$ and the associated Weil-Deligne representations, generalizing [11], and the corresponding L-packets. For this it would be helpful to know the full local Langlands correspondence for $\mathrm{GSpin}_{2g+1}(K)$, not just the part which pertains to unramified principal series representations. Since the full local

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Langlands correspondence for $\mathrm{GSpin}_{2g+1}(K)$ is almost certainly within reach by an adaptation of Arthur's work [2] on the endoscopic classification of representations, following [3], we have postponed looking into such questions until Arthur's ideas have been adapted to general spin groups. We intend to use the local results in this note to explore the connection between abelian varieties over number fields and global L-packets of automorphic representations of spin groups and general spin groups, generalizing the results of [1, §2].

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Contents

1. Abelian varieties

sec:even

In this section, we collect some useful facts about abelian varieties, especially over finite fields. Many of the attributes discussed here are isogeny invariants. We write $X \sim Y$ if X and Y are isogenous abelian varieties, and $\mathrm{End}^0(X)$ for $\mathrm{End}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$.

1.1. **Base change of abelian varieties.** Let $X/\mathbb{F}_q$ be an abelian variety of dimension $2g$. Associated to it are the characteristic polynomial $P_{X/\mathbb{F}_q}(T)$ and minimal polynomial $M_{X/\mathbb{F}_q}(T)$ of Frobenius. Then $P_{X/\mathbb{F}_q}(T) \in \mathbb{Z}[T]$ is monic of degree $2g$, and $M_{X/\mathbb{F}_q}(T)$ is the radical of $P_{X/\mathbb{F}_q}(T)$.

The isogeny class of X is completely determined by $P_{X/\mathbb{F}_q}(T)$ [17]. It is thus possible to detect from $P_{X/\mathbb{F}_q}(T)$ whether X is simple, but even easier to decide if X is isotypic, which is to say, isogenous to the self-product of a simple abelian variety. Indeed, let $\mathrm{ZEnd}^0(X) \subset \mathrm{End}^0(X)$ be the center of the endomorphism algebra of X. Then

eqzend

$$\mathrm{ZEnd}^0(X) \cong \mathbb{Q}[T]/\left(M_{X/\mathbb{F}_q}(T)\right) \tag{1.1}$$

and X is isotypic if and only if $M_{X/\mathbb{F}_q}(T)$ is irreducible. While it is possible for a simple abelian variety to become reducible after extension of scalars of the base field, isotypicality is preserved by base extension (see [9, Claim 10.8] for example).

For a monic polynomial $g(T) = \prod_{1\le j\le N}(T-\tau_j)$ and a natural number r, set $g^{(r)}(T) = \prod_{1\le j\le N}(T-\tau_j^r)$. It is not hard to check that

$$P_{X/\mathbb{F}_{q^r}}(T) = P^{(r)}_{X/\mathbb{F}_q}(T).$$

lemdimzend **Lemma 1.1.** *Suppose $X/\mathbb{F}_q$ is isotypic, and let $\mathbb{F}_{q^r}/\mathbb{F}_q$ be a finite extension. Let Y be a simple factor of $X_{\mathbb{F}_{q^r}}$. Then there exists a natural number e and some $m|r$ such that*

$$M^{(r)}_{X/\mathbb{F}_q}(T) = M_{Y/\mathbb{F}_{q^r}}(T)^e$$

and

$$\dim \mathrm{ZEnd}^0(X) = m \dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^r}}).$$

Proof. Write $X_{\mathbb{F}_{q^r}} \sim Y^n$ with Y simple. Then we have two different factorizations of $P_{X/\mathbb{F}_{q^r}}(T)$:

$$P_{X/\mathbb{F}_{q^r}}(T) = (M^{(r)}_{X/\mathbb{F}_q}(T))^d$$
$$P_{X/\mathbb{F}_{q^r}}(T) = (M_{Y/\mathbb{F}_{q^r}}(T))^e.$$

Since $M_{Y/\mathbb{F}_{q^r}}(T)$ is irreducible (and all polynomials considered here are monic), there exists some integer m such that

$$M^{(r)}_{X/\mathbb{F}_q}(T) = M_{Y/\mathbb{F}_{q^r}}(T)^m.$$

Note that

$$m = \frac{\deg M_{X/\mathbb{F}_q}(T)}{\deg M_{Y/\mathbb{F}_{q^r}}(T)} = [\mathrm{ZEnd}^0(X) : \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^r}})].$$

Let τ be a root of $M_{X/\mathbb{F}_q}(T)$. Then τ^r is a root of $M^{(r)}_{X/\mathbb{F}_q}(T)$, and thus of $M_{Y/\mathbb{F}_q}(T)$; and the inclusion of fields $\mathrm{ZEnd}^0(X_{\mathbb{F}_{q^r}}) \subseteq \mathrm{ZEnd}^0(X)$ is isomorphic to the inclusion of fields $\mathbb{Q}(\tau^r) \subseteq \mathbb{Q}(\tau)$, under (1.1). In particular, $m = [\mathbb{Q}(\tau) : \mathbb{Q}(\tau^r)]$. Since τ satisfies the equation $S^r - \tau^r$ over $\mathbb{Q}(\tau^r)$, its degree over $\mathbb{Q}(\tau^r)$ divides r. □

subsec:even 1.2. **Even abelian varieties.** Call an abelian variety $X/\mathbb{F}_q$ *even* if its characteristic polynomial is even:

$$P_{X/\mathbb{F}_q}(T) = P_{X/\mathbb{F}_q}(-T).$$

If X is simple, then it admits a unique nontrivial quadratic twist $X'/\mathbb{F}_q$. For an arbitrary $X/\mathbb{F}_q$, let $X'/\mathbb{F}_q$ be the quadratic twist associated to the cocycle

$$\begin{aligned} \mathrm{Gal}(\mathbb{F}_q) &\longrightarrow \mathrm{Aut}(X) \\ \mathrm{Fr}_q &\longmapsto [-1], \end{aligned}$$

corresponding to a nontrivial quadratic twist of all simple factors of X.

For future use, we record the following elementary observation:

lemeven **Lemma 1.2.** *Let $X/\mathbb{F}_q$ be an abelian variety. Then X is even if and only if X and X' are isogenous.*

Proof. Use the (canonical, given our construction) isomorphism $X_{\mathbb{F}_{q^2}} \cong X'_{\mathbb{F}_{q^2}}$ to identify $V_\ell X$ and $V_\ell X'$. Then one knows (see [12, p.506] for example) that $\rho_{X'/\mathbb{F}_q}(\mathrm{Fr}_q) = -\rho_{X/\mathbb{F}_q}(\mathrm{Fr}_q)$, and thus that

$$P_{X'/\mathbb{F}_q}(T) = P_{X/\mathbb{F}_q}(-T).$$

The asserted equivalence now follows from Tate's theorem. □

To a large extent, evenness of X is captured by the behavior of the center of $\mathrm{End}^0(X)$ upon quadratic base extension.

Lemma 1.3. *If $X/\mathbb{F}_q$ is even, then*

$$\dim \mathrm{ZEnd}^0(X) = 2 \dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^2}}).$$

Proof. Suppose $X/\mathbb{F}_q$ is even. Then the multiset $\{\tau_1, \cdots, \tau_{2g}\}$ of eigenvalues of Frobenius of X is stable under multiplication by -1, and in particular the set of distinct eigenvalues of Frobenius is stable under multiplication by -1. Moreover, this action has no fixed points; and thus $\{\tau_1^2, \cdots, \tau_{2g}^2\}$, the set of eigenvalues of $X/\mathbb{F}_{q^2}$, has half as many distinct elements as the original set. The claim now follows from characterization (1.1) of $\mathrm{ZEnd}^0(X)$. $\square$

The converse is almost true.

Proposition 1.4. *Suppose X is isotypic. Then X is even if and only if*

$$\dim \mathrm{ZEnd}^0(X) = 2 \dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^2}}).$$

Proof. Suppose $\dim \mathrm{ZEnd}^0(X) = 2 \dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^2}})$ and let Y be a simple factor of $X_{\mathbb{F}_{q^2}}$. By Lemma 1.1,

$$M^{(2)}_{X/\mathbb{F}_q}(T) = M_{Y/\mathbb{F}_{q^2}}(T)^2. \tag{1.2}$$

Factor the minimal polynomials of X and Y as

$$M_{X/\mathbb{F}_q}(T) = \prod_{1 \le j \le 2h} (T - \tau_j)$$
$$M_{Y/\mathbb{F}_{q^2}}(T) = \prod_{1 \le j \le h} (T - \beta_j).$$

By (1.2), we may order the roots of $M_{X/\mathbb{F}_q}(T)$ so that, for each $1 \le j \le h$, we have

$$\tau_j^2 = \tau_{h+j}^2 = \beta_j,$$

so that $\tau_{h+j} = \pm\tau_j$. In fact, $\tau_{h+j} = -\tau_j$; for otherwise, $M_{X/\mathbb{F}_q}(T)$ would have a repeated root, which contradicts the isotypicality of X. Now, $P_{X/\mathbb{F}_q}(T) = M_{X/\mathbb{F}_q}(T)^d$ for some d. The multiset of eigenvalues of Frobenius of X is thus stable under multiplication by -1, and $X/\mathbb{F}_q$ is even. $\square$

Note that evenness is an assertion about the *multiset* of eigenvalues of Frobenius, while the calculation of $\dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^e}})$ only detects the *set* of eigenvalues. Consequently, if one drops the isotypicality assumption in Proposition 1.4, it is easy to write down examples of abelian varieties which are not even but satisfy the criterion on dimensions of centers of endomorphism rings.

Example 1.5. Let $E/\mathbb{F}_q$ be an ordinary elliptic curve; then E is not isogenous to E' over $\mathbb{F}_q$ but $\mathrm{End}^0(E) \cong \mathrm{End}^0(E') \cong L$, a quadratic imaginary field. Set $X = E \times E \times E'$. Then X is not even, since $X' \cong E' \times E' \times E$, but $\mathrm{ZEnd}^0(X) \cong L \times L$ while $\mathrm{ZEnd}^0(X_{\mathbb{F}_{q^2}}) \cong L$. Thus, $X/\mathbb{F}_q$ satisfies the dimension criterion of Proposition 1.4 but is not even.

Example 1.6. Consider a supersingular elliptic curve $E/\mathbb{F}_q$, where q is an odd power of the prime p. Then $\mathrm{End}^0(E) \cong \mathbb{Q}(\sqrt{-p})$, while $\mathrm{End}^0(E_{\mathbb{F}_{q^2}})$ is the quaternion algebra ramified at p and ∞. In particular, $\mathrm{ZEnd}^0(E)$ is a quadratic imaginary field, while $\mathrm{ZEnd}^0(E_{\mathbb{F}_{q^2}}) \cong \mathbb{Q}$. Therefore, $E/\mathbb{F}_q$ is even.

Example 1.7. In contrast, if $X/\mathbb{F}_q$ is an absolutely simple ordinary abelian variety, then $\mathrm{End}^0(X) = \mathrm{End}^0(X_{\mathbb{F}_{q^2}})$. (This is a consequence of [18, Thm. 7.2], which unfortunately omits the necessary hypothesis of absolute simplicity.)

Example 1.8. Now consider an arbitrary abelian variety $X/\mathbb{F}_q$ and its preferred quadratic twist X'. Then the sum $X \times X'$ is visibly isomorphic to its own quadratic twist, and thus even.

Example 1.9. Let $X/\mathbb{F}_q$ be an abelian variety of dimension g. Suppose there is an integer $N \geq 5$, relatively prime to q, such that $X[N](\mathbb{F}_q) \cong (\mathbb{Z}/N)^{2g}$. Then X is not even. Indeed, if an abelian variety Y over a field k has maximal k-rational N-torsion for $N \geq 5$ and N invertible in k, then $\operatorname{End}^0(Y) \cong \operatorname{End}^0(Y_{\bar{k}})$ [14]. By the criterion of Lemma 1.3, if $X/\mathbb{F}_q$ satisfies the hypotheses of the present lemma, then X cannot be even.

1.3. **Abelian varieties over local fields.** Now let K be a local field with residue field $\mathbb{F}_q$ and let X/K be an abelian variety with good reduction $X_0/\mathbb{F}_q$. As in 1.2, we define a canonical quadratic twist X of X, associated to the unique nontrivial character

$$\operatorname{Gal}(\bar{K}/K) \longrightarrow \operatorname{Gal}(K^{\mathrm{unram}}/K) \longrightarrow \{[\pm 1]\} \subset \operatorname{Aut}(X).$$

Lemma 1.10. *Let X/K be an abelian variety with good reduction $X_0/\mathbb{F}_q$. The following are equivalent:*

(a) *X and X' are isogenous;*
(b) *$X_0/\mathbb{F}_q$ and $X_0'/\mathbb{F}_q$ are isogenous;*
(c) *$X_0/\mathbb{F}_q$ is even.*

Proof. By hypothesis, X spreads out to an abelian scheme $\mathcal{X}/\mathcal{O}_K$ (its Néron model) with special fibre $X_0/\mathbb{F}_q$; the automorphism $[-1] \in \operatorname{End}(X)$ extends to an automorphism of $\mathcal{X}$ and the corresponding twist $\mathcal{X}'$ has generic and special fibers X' and $(X_0)'/\mathbb{F}_q$, respectively. This compatibility explains the equivalence of (a) and (b); the equivalence of (b) and (c) is Lemma 1.2. $\square$

Call X/K *even* if X has good reduction and satisfies any of the equivalent statements in Lemma 1.10.

1.4. **Polarizations.** Let X/k be an abelian variety over an arbitrary field k. Let λ be a polarization on X, *i.e.*, a symmetric isogeny $X \to \hat{X}$ which arises from an ample line bundle on X. Fix a rational prime ℓ invertible in k. The polarization λ on X induces a nondegenerate skew-symmetric pairing $\langle\cdot,\cdot\rangle_\lambda$ on the Tate module $T_\ell X$ and on the rational Tate module $V_\ell X$. Let $\operatorname{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda)$ be the group of symplectic similitudes of $V_\ell X$ with respect to this pairing; note that $\operatorname{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda)$ comes with a representation $r_{\lambda,\ell} : \operatorname{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda) \hookrightarrow \operatorname{GL}(V_\ell X)$. Let $\rho_{X,\ell} : \operatorname{Gal}(K) \to \operatorname{GL}(V_\ell X)$ be the representation on the rational Tate module and let $\rho_{\lambda,\ell} : \operatorname{Gal}(K) \to \operatorname{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda)$ be the continuous homomorphism such that $\rho_{X,\ell} = r_{\lambda,\ell} \circ \rho_{\lambda,\ell}$.

$$\begin{tikzcd} \operatorname{Gal}(\bar{k}/k) \arrow[rr, "\rho_{X,\ell}"] \arrow[dr, "\rho_{X,\lambda,\ell}"'] & & \operatorname{GL}(V_\ell X) \\ & \operatorname{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda) \arrow[ur, "r_{\lambda,\ell}"'] & \end{tikzcd} \tag{1.3}$$

2. L-packets attached to abelian varieties

Let K be a local field. Fix a rational prime ℓ invertible in the residue field of K, and thus in K. It will be comforting, though not even remotely necessary, to fix an isomorphism $\bar{\mathbb{Q}}_\ell \cong \mathbb{C}$. We will indicate the appropriate extensions of $\rho_{X,\ell}$, $\rho_{\lambda,\ell}$ and $r_{\lambda,\ell}$ (from Section 1.4) by eliding the subscript ℓ.

2.1. **Admissible representations attached to good abelian varities.**

prop:piX **Proposition 2.1.** *Let X/K be an abelian variety of dimension g with with good reduction and let λ be a polarization on X. There is an irreducible principal series representation $\pi_{X,\lambda}$ of $\mathrm{GSpin}_{2g+1}(K)$, unique up to equivalence, such that*

$$L(z,\rho_X) = L(z,\pi_{X,\lambda}, r_\lambda).$$

Moreover, $|\ |_K^{-\frac{1}{2}} \otimes \pi_{X,\lambda}$ is unitary.

Proof. This is a very small and well-known part of the local Langlands correspondence which, in this case, matches unramified Langlands parameters taking values in $\mathrm{GSp}_{2g}(\mathbb{C})$ with unramified principal series representations of $\mathrm{GSpin}_{2g+1}(K)$. For completeness, we include the details here.

We begin by describing $L(z,\rho_X)$. By [12], the Galois representation $\rho_{X,\ell}$ is unramified and the characteristic polynomial of $\rho_{X,\ell}(\mathrm{Fr}_q)$ has rational coefficients. Accordingly, the Euler factor for $\rho_{X,\ell}$ takes the form

$$L(s,\rho_X) = \frac{(q^s)^{2g}}{P_{X_0/\mathbb{F}_q}(q^s)}.$$

Let $\{\tau_1,\dots,\tau_{2g}\}$ be the (complex) roots of $P_{X_0/\mathbb{F}_q}(T)$. Also by [12], the ℓ-adic realization $\rho_{X,\ell}(\mathrm{Fr}_q) \in \mathrm{GL}(V_\ell X)$ of the Frobenius endomorphism of X is semisimple of weight 1, so each eigenvalue satisfies $|\tau_j| = q^{1/2}$. Write $\tau_j = q^{\frac{1}{2}} e^{2\pi i\theta_j}$ for $0 \le \theta_j < 1$ and label the roots in such a way that $\theta_1 \ge \cdots \ge \theta_g \ge 0$ and $\tau_{g+j} = q\tau_j^{-1}$ for $j = 1,\dots,g$.

Set $G = \mathrm{GSpin}_{2g+1}$. Observe that GSpin_{2g+1} is dual to GSp_{2g}. Let T be a K-split maximal torus in G; let $\check{T}$ be the dual torus. Then the Lie algebra of the torus $\check{T}(\mathbb{C})$ may be identified with $X^*(T)\otimes\mathbb{C}$ through the function

$$\exp : X^*(T)\otimes\mathbb{C} \to \check{T}(\mathbb{C})$$

defined by $\check{\alpha}(\exp(x)) = e^{2\pi i\langle\check{\alpha},x\rangle}$ for each root $\check{\alpha}$ for $\check{G}$ with respect to $\check{T}$; see Appendix A. The Lie algebra of the compact part of $\check{T}(\mathbb{C})$, denoted by $\check{T}(\mathbb{C})^{\mathrm{u}}$ below, is then identified with $X^*(T)\otimes\mathbb{R}$ under exp. We pick a basis $\{e_0,\dots,e_g\}$ for $X^*(T)$ that identifies e_0 with the similitude character for $\check{G}$ and write $\{f_0,\dots,f_g\}$ for the dual basis for $X_*(T)\cong X^*(\check{T})$. Set $\theta_0 := 0$ and set $\theta := \sum_{j=0}^{g}\theta_j e_j$; note that $\theta\in X^*(T)\otimes\mathbb{R}$ so $t^{\mathrm{u}} := \exp(\theta)$ lies in $\check{T}(\mathbb{C})^{\mathrm{u}}$. Set $t := q^{\frac{1}{2}}\exp(\theta)\in\check{T}(\mathbb{C})$.

Let W_K be the Weil group for K. The L-group for T is ${}^LT = \check{T}(\mathbb{C})\times W_K$ since T is K-split. Consider the Langlands parameter

$$\phi : W'_K \to {}^LT$$

defined by $\phi(\mathrm{Fr}_q) = \rho_{X,\lambda}(\mathrm{Fr}_q) = t\times\mathrm{Fr}_q$. Let

$$\chi : T(K) \to \mathbb{C}^\times$$

be the corresponding quasicharacter. Then the unitary character

$$\chi^{\mathrm{u}} := |\ |_K^{-\frac{1}{2}} \otimes \chi$$

corresponds to the unramified Langlands parameter

$$\phi^{\mathrm{u}} : W'_K \to {}^LT$$

defined by $\phi^{\mathrm{u}}(\mathrm{Fr}_q) = t^{\mathrm{u}}\times\mathrm{Fr}_q$.

Now pick a Borel subgroup $B\subset G$ over K containing the maximal torus $T\subset G$ over K and set

$$\pi_{X,\lambda} := \mathrm{Ind}_{B(K)}^{G(K)}\,\chi.$$

Then $\pi_{X,\lambda}$ is an unramified principal series representation of $G(K)$. The R-group for $\pi_{X,\lambda}$ (see [8] and [13]), which governs the decomposition of this principal series representation into irreducible representations, is the component group

$$\pi_0(Z_{\check{G}(\mathbb{C})}(t)) = Z_{\check{G}(\mathbb{C})}(t)/Z_{\check{G}(\mathbb{C})}(t)^0.$$

Since the derived group of $\check{G} = \mathrm{GSpin}_{2g+1}$ is simply connected, this component group is trivial and $\pi_{X,\lambda}$ is irreducible. In the same way, the unitary character $\chi^{\mathrm{u}} : K^\times \to \mathbb{C}^\times$ determines the irreducible principal series representation

$$\pi^{\mathrm{u}}_{X,\lambda} := \mathrm{Ind}_{B(K)}^{G(K)} \chi^{\mathrm{u}}.$$

This admissible representation π^{u}_λ is unitary and enjoys

$$\pi^{\mathrm{u}}_{X,\lambda} = |\ |_K^{-\frac{1}{2}} \otimes \pi_{X,\lambda},$$

as promised.

Having identified the irreducible principal series representation $\pi_{X,\lambda}$ of $\mathrm{GSpin}_{2g}(K)$ attached to (X,λ), we turn to the L-function $L(s,\pi_{X,\lambda},r_\lambda)$. For this it will be helpful to go back and say a few words about the representation $r_{\lambda,\ell} : \mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda) \hookrightarrow \mathrm{GL}(V_\ell X)$.

Let S be a maximal torus in $\mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda)$ containing $\rho_{X,\ell}(\mathrm{Fr}_q)$ and let S' be a maximal torus in $\mathrm{GL}(V_\ell X)$ containing $r_{\lambda,\ell}(S)$. Let F_ℓ be the splitting extension of S' in $\bar{\mathbb{Q}}_\ell$; observe that this is the splitting extension of $P_{X_0/\mathbb{F}_q}(T) \in \mathbb{Q}[T]$ in $\bar{\mathbb{Q}}_\ell$. Passing from $\mathbb{Q}_\ell$ to F_ℓ, we may choose bases $\{f_0, f_1, f_2, \dots, f_g\}$ for $X^*(S)$ and $\{f'_1, f'_2, \dots, f'_{2g}\}$ for $X^*(S')$ and such that map $X^*(S') \twoheadrightarrow X^*(S)$ induced by the representation $r_{\lambda,\ell}$ is given by

$$X^*(S') \ \to \ X^*(S) \qquad f'_j \mapsto f_j,\ f'_{g+j} \mapsto f_0 - f_{g-j+1}, \quad j = 1, \dots, g. \tag{2.1}$$

Note that this determines a basis for $V_\ell X \otimes_{\mathbb{Q}_\ell} F_\ell$.

Passing from F_ℓ to $\mathbb{C}$ we have now identified a basis for $V_\ell X \otimes_{\mathbb{Q}_\ell} \mathbb{C}$ which defines

$$\mathrm{GSp}(V_\ell X \otimes_{\mathbb{Q}_\ell} \mathbb{C}, \langle\cdot,\cdot\rangle_\lambda) \cong \mathrm{GSp}_{2g}(\mathbb{C})$$

inducing $S \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong \check{T}$ and also gives

$$\mathrm{GL}(V_\ell X \otimes_{\mathbb{Q}_\ell} \mathbb{C}) \cong \mathrm{GL}_{2g}(\mathbb{C}).$$

Now (1.3) extends to

$$\begin{tikzcd} \mathrm{Gal}(\bar{K}/K) \arrow[rr, "\rho_X"] \arrow[dr, "\rho_{X,\lambda}"'] & & \mathrm{GL}_{2g}(\mathbb{C}) \\ & \mathrm{GSp}_{2g}(\mathbb{C}) \arrow[ur, "r_\lambda"'] & \end{tikzcd} \tag{2.2}$$

It follows immediately that

$$L(s,\pi_{X,\lambda},r_\lambda) = \prod_{i=1}^{2g} \frac{1}{1-\tau_i q^{-s}} = \prod_{i=1}^{2g} \frac{q^s}{q^s - \tau_i} = \frac{(q^s)^{2g}}{P_{X_0/\mathbb{F}_q}(q^s)} = L(s,\rho_X),$$

concluding the proof of Proposition 2.1. $\square$

2.2. **Reducibility.** In this section we show how to recognize when X/K is even through a simple property of the admissible representation $\pi_{X,\lambda}$ of $\mathrm{GSpin}_{2g+1}(K)$, with $g = \dim X$.

Theorem 2.2. *Let X/K be an abelian variety of dimension g with good reduction and let λ be a polarization on X. The restriction of $\pi_{X,\lambda}$ from $\mathrm{GSpin}_{2g+1}(K)$ to $\mathrm{Spin}_{2g+1}(K)$ is reducible if and only if X is even.*

Proof. Recall notation $G = \mathrm{GSpin}_{2g+1}$. Then $G_{\mathrm{der}} = \mathrm{Spin}_{2g+1}$ with dual group $\check{G}_{\mathrm{ad}} = \mathrm{PGSp}_{2g}$. As in the proof of Proposition 2.1, let $t \in \check{T}(\mathbb{C})$ be the image of Fr_q under $\rho_{X,\lambda}$. The restriction of $\pi_{X,\lambda}$ from $G(K) = \mathrm{GSpin}_{2g+1}(K)$ to $G_{\mathrm{der}}(K) = \mathrm{Spin}_{2g+1}(K)$ decomposes into irreducible representations indexed by the component group $\pi_0(Z_{\check{G}_{\mathrm{ad}}}(t_{\mathrm{ad}}))$, where t_{ad} is the image of t under $\check{T} \to \check{T}_{\mathrm{ad}}$. Indeed, the irreducible representations of $G_{\mathrm{der}}(K)$ that arise in this way are precisely the irreducible representations appearing in $\mathrm{Ind}_{B_{\mathrm{der}}(K)}^{G_{\mathrm{der}}(K)} \chi_{\mathrm{der}}$, where $B_{\mathrm{der}}(K)$ is a Borel containing $T_{\mathrm{der}}(K)$ and χ_{der} is the unramified quasicharacter of $T_{\mathrm{der}}(K)$ corresponding to $t_{\mathrm{ad}} \in \check{T}_{\mathrm{ad}}(\mathbb{C})$. The R-group for this unramified prinicipal series representation is

$$\pi_0(Z_{\check{G}_{\mathrm{ad}}}(t_{\mathrm{ad}})).$$

By Lemma A.2, this group is either trivial or a group of order 2, so either $\pi_{X,\lambda}|_{G_{\mathrm{der}}(K)}$ is irreducible or contains two irreducible admissible representations; also by Lemma A.2, the latter case occurs if and only if the characteristic polynomial $P_{X_0/\mathbb{F}_q}(T)$ is even, in which case X/K itself is even (Lemma 1.10). □

2.3. **L-packet interpretation.** In this section we show how to recognize even abelian varieties over local fields through associated L-packets.

As discussed in Section 1.4, every polarized abelian variety (X, λ) over K determines an ℓ-adic Galois representation $\rho_{X,\lambda,\ell} : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda)$. Let $\phi_{X,\lambda,\ell} : W_K' \to \mathrm{Gal}(\bar{K}/K) \to \mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda)$ be the Weil-Deligne homomorphism obtained by applying [6, Thm 8.2] to $\rho_{X,\lambda,\ell}$. We note that ${}^L G = \check{G}(\mathbb{C}) \rtimes W_K = \mathrm{GSp}_{2g}(\mathbb{C}) \times W_K$. Let

$$\phi_{X,\lambda} : W_K' \to \mathrm{Gal}(\bar{K}/K) \to {}^L G$$

be the admissible homomorphism determined by $\phi_{X,\lambda,\ell}$ and the basis for $V_\ell X \otimes_{\mathbb{Q}_\ell} \mathbb{C}$ identified in the proof of Proposition 2.1. The equivalence class of $\phi_{X,\lambda}$ in $\Phi(G/K)$ is the Langlands parameter for the polarized abelian variety (X, λ) over K. We remark that this recipe is valid for all polarized abelian varieties over K, not just those of good reduction. But here we are interested in the case when X has good reduction, which we now assume. In this case $\rho_{X,\lambda}$ is unramified so the Langlands parameter $\phi_{X,\lambda}$ is both trivial on the inertia subgroup I_K of W_K in W_K' and has trivial local monodromy operator.

Although the full local Langlands correspondence for GSpin_{2g+1} is not yet known, the part which pertains to unramified principal series representations is, allowing us to consider the L-packet $\Pi_{X,\lambda}$ for the Langlands parameter $\phi_{X,\lambda}$. Indeed, we have seen that this L-packet contains the equivalence class of $\pi_{X,\lambda}$, only.

Theorem 2.2 shows that we can detect when X is K-isogenous to its twist over the quadratic unramified extension of K by restricting $\pi_{X,\lambda}$ from $G(K)$ to $G_{\mathrm{der}}(K)$. On the Langlands parameter side, this restriction corresponds to post-composing $\phi_{X,\lambda}$ with ${}^L G \to {}^L G_{\mathrm{ad}}$. Let $\phi_{X,\lambda}^{\mathrm{der}}$ be the Langlands parameter for G_{der}/K defined by the diagram below and let $\Pi_{X,\lambda}^{\mathrm{der}}$ be the corresponding

L-packet.

phiad (2.3)
$$\begin{tikzcd} W'_K \arrow[rr, "\phi_{X,\lambda}"] \arrow[dr, dotted, "\phi^{\mathrm{der}}_{X,\lambda}"'] & & {}^L G \arrow[dl] \\ & {}^L G_{\mathrm{ad}} & \end{tikzcd}$$

cor:L **Corollary 2.3.** *Let X/K be an abelian variety of dimension g with good reduction and let λ be a polarization on X. The L-packet $\Pi^{der}_{X,\lambda}$ for $\mathrm{Spin}_{2g+1}(K)$ has cardinality 2 exactly when X is even; otherwise, it has cardinality 1.*

Proof. This follows directly from the fact that the R-group for any representation in the restriction of $\pi_{X,\lambda}$ to $G_{\mathrm{der}}(K)$ coincides with the Langlands component group attached to $\phi^{\mathrm{der}}_{X,\lambda}$. (See [4] for more instances of this coincidence.) Namely, equivalence classes of representations that live in $\Pi^{\mathrm{der}}_{X,\lambda}$ are parameterized by irreducible representations of the group

$$\mathcal{S}_{\phi^{\mathrm{der}}_{X,\lambda}} := Z_{\check{G}_{\mathrm{ad}}}(\phi^{\mathrm{der}}_{X,\lambda})/Z_{\check{G}_{\mathrm{ad}}}(\phi^{\mathrm{der}}_{X,\lambda})^0\ (Z\check{G}_{\mathrm{ad}})^{W_K}.$$

Since G_{der} is K-split, the action of W_K on $\check{G}_{\mathrm{ad}}$ is trivial, and since $\phi^{\mathrm{der}}_{X,\lambda}$ is unramified, $Z_{\check{G}_{\mathrm{ad}}}(\phi^{\mathrm{der}}_{X,\lambda}) = Z_{\check{G}_{\mathrm{ad}}}(t_{\mathrm{ad}})$, where $t_{\mathrm{ad}} = \phi^{\mathrm{der}}_{X,\lambda}(\mathrm{Fr}_q)$; thus,

$$\mathcal{S}_{\phi^{\mathrm{der}}_{X,\lambda}} = \pi_0(Z_{\check{G}_{\mathrm{ad}}}(\tau_{\mathrm{ad}})),$$

which precisely the R-group for $\pi_{X,\lambda}|_{G_{\mathrm{der}}(K)}$, calculated in Theorem 2.2. $\square$

app:cpt

Appendix A. Component group calculations

The lemma used in the proof of Theorem 2.2 follows from elementary facts about connected reductive groups over algebraically closed fields and some peculiarities of symplectic groups. For completeness, in this appendix we recall what is needed for present purposes.

Let H be a connected, reductive algebraic group over an algebraically closed field which, without loss for our application, we take to be $\mathbb{C}$. Fix a maximal torus $S \subseteq H$. Let R be the root system for $S \subseteq H$ and let W be the Weyl group for R.

Let X (resp. P, Q) be the character lattice (resp. weight lattice, root lattice) for H with respect to S; let $X^\vee$ (resp. $P^\vee$, $Q^\vee$) be the co-character lattice (resp. co-weight lattice, co-root lattice) for H with respect to S. The function

$$\exp : X^\vee \otimes \mathbb{C} \to S,$$

defined by $a(\exp(x)) = e^{2\pi i \langle a,x\rangle}$ for $a \in R$, allows us to identify $\mathrm{Lie}\, S$ with $X^\vee \otimes \mathbb{C}$. Under this identification, the Lie algebra of the compact subtorus $S_{\mathrm{cpt}} \subseteq S$ may be identified with $X^\vee \otimes \mathbb{R}$, henceforth denoted by V.

lem:xtos **Lemma A.1.** *If $v \in V$ and $s = \exp(v)$ then*

$$\pi_0(Z_H(s)) \cong \{\gamma \in P^\vee/Q^\vee \mid \gamma(v) = v\},$$

for a canonical action of $P^\vee/Q^\vee$ on V given below. In particular, $\pi_0(Z_H(s)) \subseteq P^\vee/Q^\vee$, which is a finite abelian group.

Proof. By [15, §3.5, Prop. 4] (see also [7, §2.2, Theorem]), $Z_H(s)$ is a reductive group with root system $R_s := \{a \in R \mid a(s) = 1\}$ and component group $W(s)/W_{Z_H(s)^0}$. We wish to compute this group.

Adapting [5, VI,§2], let $W_{\text{aff}} := Q^\vee \rtimes W$ be the affine Weyl group for H and let $W_{\text{ext}} := P^\vee \rtimes W$ be the extended affine Weyl group. Then

$$1 \to W_{\text{aff}} \to W_{\text{ext}} \to P^\vee/Q^\vee \to 0 \tag{A.1}$$

is a short exact sequence defining W_{ext} as a semidirect product of the Coxeter group W_{aff} by $P^\vee/Q^\vee$. Since $P^\vee/Q^\vee$ is a quotient of two lattices of equal rank, it is a finite abelian group; in fact, $P^\vee/Q^\vee$ coincides with the fundamental group $\pi_1(H)$ of H (see [16, p. 45] for a table of these finite abelian groups by type). By [5, VI,§2.4, Cor.], the miniscule coweights for H determine a set of representatives for $P^\vee/Q^\vee$. We will see that $W(s)/W_{Z_H(s)}$ is the fixator of v in $P^\vee/Q^\vee$, for $\exp(v) = s$. To do this we must describe the action of $P^\vee/Q^\vee$ on V. First, however note that $w(s) = s$ if and only if $w(v) - v \in P^\vee$.

Assuming henceforth that Q is non-empty, choose a basis $\Delta = \{a_1, \ldots, a_n\}$ for Q. This determines an alcove in V according to the rule

$$C = \{v \in V \mid \langle a_i, v\rangle > 0, 0 \leq i \leq n\}$$

where a_0 is the affine root $1 - \tilde{a}$ and where $\tilde{a}$ is the longest root with respect to Δ; see [5, VI, §2.3]. The closure $\bar{C}$ of C is a fundamental domain for the action of W_{aff} on V. The affine Weyl group W_{aff} acts freely and transitively on the set of alcoves in V. The extended affine Weyl group W_{ext} acts transitively on the set of alcoves, but generally not freely. Since miniscule coweights for H determine a set of representatives for $P^\vee/Q^\vee$, and since each such coweight may be identified with a vertex of $\bar{C}$ (not all vertices arise this way), we have

$$\{w \in W_{\text{ext}} \mid w(C) = C\} \cong P^\vee/Q^\vee, \tag{A.2}$$

canonically.[1]

Now we see that the group

$$W_{\text{aff}}(v) := \{w \in W_{\text{aff}} \mid w(v) = v\}$$

acts simply and transitively on the set of alcoves for which v lies in the boundary. Set

$$W_{\text{ext}}(v) := \{w \in W_{\text{ext}} \mid w(v) = v\}$$

and, using (A.2), set

$$(P^\vee/Q^\vee)(v) := \{\gamma \in P^\vee/Q^\vee \mid \gamma(v) = v\}.$$

The inclusions $W_{\text{aff}}(v) \hookrightarrow W_{\text{aff}}$, $W_{\text{ext}}(v) \hookrightarrow W_{\text{ext}}$ and $(P^\vee/Q^\vee)(x) \hookrightarrow P^\vee/Q^\vee$, together with (A.1), give a short exact sequence

$$1 \to W_{\text{aff}}(v) \to W_{\text{ext}}(v) \to (P^\vee/Q^\vee)(v) \to 0. \tag{A.3}$$

The image of $W_{\text{ext}}(v)$ under the projection $W_{\text{ext}} \to W$ is $W(s)$ while the image of $W_{\text{aff}}(v)$ under the projection $W_{\text{aff}} \to W$ is $W_{Z_H(s)}$; see, for example, the proof of [10, Prop. 2.1]. Recalling that $w(s) = s$ if and only if $w(v) - v \in P^\vee$, and also using the fact that $W_{\text{ext}}(v)$ is finite and $P^\vee$ is a lattice, we see that these projections are group isomorphisms (again, see [10, Prop. 2.1]). Thus, when applied to (A.3), projection to W gives a short exact sequence

$$1 \to W_{Z_H(s)^0} \to W(s) \to (P^\vee/Q^\vee)(v) \to 0.$$

This concludes the proof of Lemma A.1. ☐

[1] This isomorphism is also described (very nicely!) in [10, §2.2] under the hypothesis that H has trivial centre; to treat the more general case here we replace $\tilde{W} := X^\vee \rtimes W$ with $W_{\text{ext}} := P^\vee \rtimes W$ and follow Bourbaki.

TABLE 1. Based root data for GSpin_{2g+1}, Spin_{2g+1} and SO_{2g+1}.

semisimple, simply connected		**Type:** B_g		adjoint
$G_{\mathrm{der}} = \mathrm{Spin}_{2g+1}$ $T_{\mathrm{der}} = \mathbb{G}_{\mathbf{m}}^g$ $Z(G_{\mathrm{der}}) = \mu_2$	$\rightarrowtail$ $\rightarrowtail$ $\rightarrowtail$	$G = \mathrm{GSpin}_{2g+1}$ $T = \mathbb{G}_{\mathbf{m}}^{g+1}$ $Z(G) = \mathbb{G}_{\mathbf{m}}$	$\twoheadrightarrow$ $\twoheadrightarrow$ $\twoheadrightarrow$	$G_{\mathrm{ad}} = \mathrm{SO}_{2g+1}$ $T_{\mathrm{ad}} = \mathbb{G}_{\mathbf{m}}^g$ $Z(G_{\mathrm{ad}}) = 1$
$X^*(T_{\mathrm{der}}) = \langle e_1, \dots, e_g \rangle$ $R_{\mathrm{der}} := R(G_{\mathrm{der}}, T_{\mathrm{der}})$ $= \langle \alpha'_1, \dots, \alpha'_g \rangle$ $\alpha'_1 = e_1 - e_2$ $\alpha'_1 = e_2 - e_3$ $\vdots$ $\alpha'_{g-1} = e_{g-1} - e_g$ $\alpha'_g = e_g$	$0 \leftarrowtail e_0$	$X^*(T) = \langle e_0, e_1, \dots, e_g \rangle$ $R := R(G, T)$ $= \langle \alpha_1, \dots, \alpha_g \rangle$ $\alpha_1 = e_1 - e_2$ $\alpha_2 = e_2 - e_3$ $\vdots$ $\alpha_{g-1} = e_{g-1} - e_g$ $\alpha_g = e_g$	$\leftarrowtail$	$X^*(T_{\mathrm{ad}}) = \langle \alpha_1, \dots, \alpha_g \rangle$ $R_{\mathrm{ad}} := R(G_{\mathrm{ad}}, T_{\mathrm{ad}})$ $= \langle \alpha_1, \dots, \alpha_g \rangle$
$X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle = \mathbb{Z}/2\mathbb{Z}$		$X^*(T)/\langle R \rangle = \mathbb{Z}$		$X^*(T_{\mathrm{ad}})/\langle R_{\mathrm{ad}} \rangle = 0$
adjoint		**Type:** C_g		semisimple, simply connected
$\check{G}_{\mathrm{ad}} = \mathrm{PGSp}_{2g}$ $\check{T}_{\mathrm{ad}} = \mathbb{G}_{\mathbf{m}}^g$ $Z(\check{G}_{\mathrm{ad}}) = 1$	$\twoheadleftarrow$ $\twoheadleftarrow$ $\twoheadleftarrow$	$\check{G} = \mathrm{GSp}_{2g}$ $\check{T} = \mathbb{G}_{\mathbf{m}}^{g+1}$ $Z(\check{G}) = \mathbb{G}_{\mathbf{m}}$	$\leftarrowtail$ $\leftarrowtail$ $\leftarrowtail$	$\check{G}_{\mathrm{der}} = \mathrm{Sp}_{2g}$ $\check{T}_{\mathrm{der}} = \mathbb{G}_{\mathbf{m}}^g$ $Z(\check{G}_{\mathrm{der}}) = \mu_2$
$X^*(\check{T}_{\mathrm{ad}}) = \langle \check{\alpha}_1, \dots, \check{\alpha}_g \rangle$ $\check{R}_{\mathrm{ad}} := R(\check{G}_{\mathrm{ad}}, \check{T}_{\mathrm{ad}})$ $= \langle \check{\alpha}_1, \dots, \check{\alpha}_g \rangle$	$\rightarrowtail$	$X^*(\check{T}) = \langle f_0, f_1, \dots, f_g \rangle$ $\check{R} := R(\check{G}, \check{T})$ $= \langle \check{\alpha}_1, \dots, \check{\alpha}_g \rangle$ $\check{\alpha}_1 = f_1 - f_2$ $\check{\alpha}_2 = f_2 - f_3$ $\check{\alpha}_{g-1} = f_{g-1} - f_g$ $\check{\alpha}_g = 2f_g - f_0$	$f_0 \mapsto 0$ $\vdots$	$X^*(\check{T}_{\mathrm{der}}) = \langle f_1, \dots, f_g \rangle$ $\check{R}_{\mathrm{der}} := R(\check{G}_{\mathrm{der}}, \check{T}_{\mathrm{der}})$ $= \langle \check{\alpha}'_1, \dots, \check{\alpha}'_g \rangle$ $\check{\alpha}'_1 = f_1 - f_2$ $\check{\alpha}'_2 = f_2 - f_3$ $\vdots$ $\check{\alpha}'_{g-1} = f_{g-1} - f_g$ $\check{\alpha}'_g = 2f_g$
$X^*(\check{T}_{\mathrm{ad}})/\langle \check{R}_{\mathrm{ad}} \rangle = 0$		$X^*(\check{T})/\langle \check{R} \rangle = \mathbb{Z}$		$X^*(\check{T}_{\mathrm{der}})/\langle \check{R}_{\mathrm{der}} \rangle = \mathbb{Z}/2\mathbb{Z}$

Lemma A.2. *Suppose $t \in \mathrm{GSp}_{2g}(\mathbb{C})$ is semisimple and all eigenvalues have complex norm 1. Let s be the image of t under $\mathrm{GSp}_{2g}(\mathbb{C}) \to \mathrm{PGSp}_{2g}(\mathbb{C})$. Then $\pi_0(Z_{\mathrm{PGSp}_{2g}(\mathbb{C})}(s)) = \mathbb{Z}/2\mathbb{Z}$ if and only if the characteristic polynomial of t is even; otherwise, $Z_{\mathrm{PGSp}_{2g}(\mathbb{C})}(s)$ is connected.*

Proof. We wish to use Lemma A.1 in the case $H = \check{G}_{\mathrm{ad}}(\mathbb{C}) = \mathrm{PGSp}_{2g}(\mathbb{C})$. The root datum for PGSp_{2g} can be read from Table 1; note that PGSp_{2g} is the dual group to Spin_{2g+1}. Since PGSp_{2g} is adjoint, its coweight lattice $P^\vee$ coincides with its cocharacter lattice $X^\vee$, which is $X^*(T_{\mathrm{der}})$ in the notation of Table 1, and its coroot lattice $Q^\vee$ is $\langle R_{\mathrm{der}} \rangle$. Thus, with reference to Lemma A.1,

$$P^\vee/Q^\vee = X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle = \mathbb{Z}/2\mathbb{Z}.$$

Without loss of generality, we may suppose $t \in \check{T}(\mathbb{C})$ so $s \in \check{T}_{\mathrm{ad}}(\mathbb{C})$ (see Table 1 again). Lemma A.1 now shows that

$$Z_{\mathrm{PGSp}_{2g}(\mathbb{C})}(s)/Z_{\mathrm{PGSp}_{2g}(\mathbb{C})}(s)^0 \hookrightarrow \mathbb{Z}/2\mathbb{Z};$$

accordingly, the group $\pi_0(Z_{\mathrm{PGSp}_{2g}(\mathbb{C})}(s))$ is either trivial or order 2.

Again with reference to the notation of Lemma A.1 and Table 1, for $H = \check{G}_{\mathrm{ad}}(\mathbb{C})$ we have $X = X_*(T_{\mathrm{ad}})$, $X^\vee = X^*(T_{\mathrm{der}})$ and $S = \check{T}_{\mathrm{ad}}(\mathbb{C})$, so $\mathrm{Lie}\, S = X^*(T_{\mathrm{der}}) \otimes \mathbb{C}$, $\mathrm{Lie}\, S_{\mathrm{cpt}} = X^*(T_{\mathrm{der}}) \otimes \mathbb{R} = V$.

Pick $x \in X^*(T) \otimes \mathbb{R}$ so that $\exp(x) = t$. Let x_{der} be the image of x under $X^*(T) \to X^*(T_{\mathrm{der}})$; then $\exp(x_{\mathrm{der}}) = t_{\mathrm{ad}}$. Let $\{\varpi_1, \dots, \varpi_g\}$ be the basis of weights for $X^\vee = X^*(T_{\mathrm{der}})$ dual to the basis $\check{R} = \{\check{\alpha}_1, \dots, \check{\alpha}_g\}$ for $X^*(\check{T}_{\mathrm{ad}}) = X_*(T_{\mathrm{der}})$; set $\varpi_0 = 0$. The alcove determined by $\check{R}$ is

$$C = \{v \in V \mid \langle \check{\alpha}_j, v\rangle > 0,\ 0 \le j \le g\}.$$

The closure $\bar{C}$ of C is the convex hull of the vertices $\{v_0, v_1, \dots, v_g\}$ defined by $v_j = \frac{1}{b_j}\varpi_j$, where $b_0 = 1$ and the other integers b_j are determined by the longest root in $\check{R}_{\mathrm{ad}} = \check{R}$ according to $\check{\alpha} = \sum_{j=1}^{g} b_j \check{\alpha}_j$; in the case at hand the longest root is $\check{\alpha} = 2\check{\alpha}_1 + 2\check{\alpha}_2 + \cdots + 2\check{\alpha}_{g-1} + \check{\alpha}_g$ so $b_1 = 2$, $\dots b_{g-1} = 2$, $b_g = 1$. Note that exactly two vertices in $\{v_0, v_1, \dots, v_g\}$ are hyperspecial: v_0 and v_g.

Since W_{ext} acts transitively on the alcoves in V and since $\exp : V \to S$ is W_{ext} invariant, we may now suppose $x_{\mathrm{der}} \in \bar{C}$. Set $v = x_{\mathrm{der}}$. Express $v \in V$ in the basis of weights for $X^\vee = X^*(T_{\mathrm{der}})$:

$$v = \sum_{j=1}^{g} x_j \varpi_j; \tag{A.4}$$

note that the coefficients in this expansion are precisely the root values $x_j = \check{\alpha}_j(v)$. Then $v \in \bar{C}$ exactly means $x_j \ge 0$. Set $b_0 = 1$ and define $x_0 \ge 0$ so that $\sum_{j=0} b_j x_j = 1$; in other words

$$v = \sum_{j=0}^{g} x_j \varpi_j, \qquad x_0 + 2x_1 + \cdots + 2x_{g-1} + x_g = 1.$$

Under the isomorphism (A.2), the non-trivial element of $P^\vee/Q^\vee$ corresponds to $\rho \in W_{\mathrm{ext}}$ defined by $v_j \mapsto v_{g-j}$ for $j = 0, \dots, g$. In terms of the fundamental weights $\{\varpi_0, \varpi_1, \dots, \varpi_g\}$ this affine transformation is defined by $\varpi_j \mapsto \varpi_{g-j}$ for $j = 0, \dots, g$. Thus, $(P^\vee/Q^\vee)(v)$ is non-trivial if and only if $\rho(v) = v$, which is so say,

$$x_j \quad \ge \quad 0, \quad j = 1, \dots, g \tag{A.5}$$

and

$$\begin{aligned} x_1 + \cdots + x_{g-1} + x_g &= \frac{1}{2} \\ x_j &= x_{g-j}, \quad j = 1, \dots, g-1. \end{aligned}$$

Now we translate the conditions above into conditions on the eigenvalues of $t = \exp(x)$. To do this, we pass from root values $x_j = \langle \check{\alpha}_j, x\rangle$ to character values $y_j := \langle f_j, x\rangle$. Again using Table 1 we see that the conditions above are equivalent to

$$y_1 \ge y_2 \ge \cdots \ge y_g \quad \ge \quad \frac{1}{2} y_0 \tag{A.6}$$

and

$$\begin{aligned} y_1 + y_g &= \frac{1}{2} + y_0 \\ y_j - y_{j+1} &= y_{g-j} - y_{g-j+1}, \quad j = 1, \dots, g-1. \end{aligned}$$

When combined, these last two conditions take a very simple form:

$$y_0 - y_j \quad = \quad \frac{1}{2} + y_{g-j+1}, \quad j = 1, \dots, g-1. \tag{A.7}$$

In order to finish the proof of Lemma A.2 we consider the characteristic polynomial of $r(t)$. Observe that $r(t) = r(\exp(x)) = \exp(dr(x))$, where $dr : X^*(T) \to X^*(\mathbb{G}_{\mathbf{m}}^{2g})$ is given by (2.1). Set $t_j = e^{2\pi i y_j}$ for $j = 0, \ldots, g$. Then the (A.7) is equivalent to

$$t_0 t_j^{-1} = -t_{g-j+1}, \quad j = 1, \ldots, g-1. \tag{A.8}$$

The characteristic polynomial of $r(t)$ is

$$P_{r(t)}(T) := \prod_{j=1}^{g} (T - t_j) \prod_{j=1}^{g} (T - t_0 t_j^{-1}). \tag{A.9}$$

When combined with (A.8), it is clear that $P_{r(t)}(T)$ is even:

$$\begin{aligned} P_{r(t)}(T) &= \prod_{j=1}^{g} (T - t_j) \prod_{j=1}^{g} (T + t_{g-j+1}), \quad \text{(A.8)} \\ &= \prod_{j=1}^{g} (T - t_j) \prod_{i=1}^{g} (T + t_i), \quad j \mapsto g - j + 1 \\ &= \prod_{j=1}^{g} (T^2 - t_j^2). \end{aligned}$$

We have now seen that if $\pi_0(Z_{\mathrm{PGSp}_{2g}(\mathbb{C})}(s))$ is non-trivial, then $P_{r(t)}(T)$ is even. To see the converse, suppose $P_{r(t)}(T)$ (A.9) is even. Without loss of generality, we may assume the similitude factor t_0 is trivial. Then, after relabelling if necessary, the *symplectic* characteristic polynomial $P_{r(t)}(T)$ is even if and only if it takes the form $P_{r(t)}(T) = \prod_{j=1}^{g} (T^2 - r_j^2)$, with $r_j^{-1} = -r_{\sigma(j)}$ for some permutation σ of $\{1, \ldots, g\}$. Since the roots are the eigenvalues of t, which are unitary by hypothesis, we can order them by angular components, as in (A.6), while replacing σ with the permutation $j \mapsto g - j + 1$, thus bringing us back to (A.7). This concludes the proof of Lemma A.2. $\square$

References

[1] U. K. Anandavardhanan and Dipendra Prasad, *On the* SL(2) *period integral*, Amer. J. Math. **128** (2006), no. 6, 1429–1453.

[2] James Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[3] ———, *Automorphic representations of* GSp(4), Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 65–81.

[4] Dubravka Ban and David Goldberg, *R-groups and parameters*, Pacific J. Math. **255** (2012), no. 2, 281–303.

[5] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[6] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 501–597. Lecture Notes in Math., Vol. 349.

[7] James E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, vol. 43, American Mathematical Society, Providence, RI, 1995.

[8] C. David Keys, *L-indistinguishability and R-groups for quasisplit groups: unitary groups in even dimension*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 1, 31–64.

Oort:2005survey [9] Frans Oort, *Abelian varieties over finite fields*, Higher-dimensional geometry over finite fields (Dmitry and Tschinkel Kaledin Yuri, ed.), NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., vol. 16, IOS, Amsterdam, 2008, pp. 123–188.

Reeder [10] Mark Reeder, *Torsion automorphisms of simple Lie algebras*, Enseign. Math. (2) **56** (2010), no. 1-2, 3–47.

Rohrlich [11] David E. Rohrlich, *Elliptic curves and the Weil-Deligne group*, Elliptic curves and related topics, CRM Proc. Lecture Notes, vol. 4, Amer. Math. Soc., Providence, RI, 1994, pp. 125–157.

Serre-Tate:good [12] Jean-Pierre Serre and John Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.

Silberger [13] Allan J. Silberger, *Introduction to harmonic analysis on reductive p-adic groups*, Mathematical Notes, vol. 23, Princeton University Press, Princeton, N.J., 1979. Based on lectures by Harish-Chandra at the Institute for Advanced Study, 1971–1973.

Silverberg [14] A. Silverberg, *Fields of definition for homomorphisms of abelian varieties*, J. Pure Appl. Algebra **77** (1992), no. 3, 253–262.

Steinberg [15] Robert Steinberg, *Conjugacy classes in algebraic groups*, notes by Vinay V. Deodhar, Lecture Notes in Mathematics, vol. 366, Springer–Verlag, Berlin, 1974.

Steinberg:Yale [16] ———, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968. Notes prepared by John Faulkner and Robert Wilson.

Tate:endff [17] John Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.

Waterhouse [18] William C. Waterhouse, *Abelian varieties over finite fields*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 521–560.

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