COUNTING ABELIAN VARIETIES OVER FINITE FIELDS VIA FROBENIUS DENSITIES

JEFFREY D. ACHTER, S. ALI ALTUĞ, LUIS GARCIA, AND JULIA GORDON

ABSTRACT. Let $[X, \lambda]$ be a principally polarized abelian variety over a finite field with commutative endomorphism ring; further suppose that either X is ordinary or the field is prime. Motivated by an equidistribution heuristic, we introduce a factor $\nu_v([X, \lambda])$ for each place v of $\mathbb{Q}$, and show that the product of these factors essentially computes the size of the isogeny class of $[X, \lambda]$.

The derivation of this mass formula depends on a formula of Kottwitz and on analysis of measures on the group of symplectic similitudes, and in particular does not rely on a calculation of class numbers.

1. INTRODUCTION

Let $[X, \lambda] \in \mathcal{A}_g(\mathbb{F}_q)$ be a principally polarized g-dimensional abelian variety over the finite field $\mathbb{F}_q = \mathbb{F}_{p^e}$. Its isogeny class $I([X, \lambda], \mathbb{F}_q)$ is finite; our goal is to understand the (weighted by automorphism group) cardinality $\widetilde{\#} I([X, \lambda], \mathbb{F}_q)$.

A random matrix heuristic might suggest the following. Let $f_{X/\mathbb{F}_q}(T)$ be the characteristic polynomial of Frobenius of X. It is well-known that $f_{X/\mathbb{F}_q}(T) \in \mathbb{Z}[T]$. Following Gekeler [Gek03], for a rational prime $\ell \nmid p \operatorname{disc}(f)$, one can define a number

$$\nu_\ell([X, \lambda], \mathbb{F}_q) = \lim_{n \to \infty} \frac{\#\{\gamma \in \mathrm{GSp}_{2g}(\mathbb{Z}_\ell/\ell^n) : \operatorname{charpoly}_\gamma(T) = f_{X/\mathbb{F}_q}(T) \bmod \ell^n\}}{\#\,\mathrm{GSp}_{2g}(\mathbb{Z}_\ell/\ell^n)/(\ell-1)\ell^{n-1}\ell^{gn}}. \tag{1.1}$$

For $\ell \nmid p \operatorname{disc}(f)$, in which case the conjugacy class is determined by the characteristic polynomial (cf. Lemma 3.1), we interpret $\nu_\ell[X, \lambda]$ as the deviation of the size of the conjugacy class with characteristic polynomial $f_{X/\mathbb{F}_q}(T)$ from the average size of a conjugacy class in $\mathrm{GSp}(\mathbb{Z}_\ell)$.

For $\ell \mid \operatorname{disc}(f_{X/\mathbb{F}_q}(T))$, since the characteristic polynomial need not determine a unique conjugacy class in $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$, a slightly more involved definition of $\nu_\ell[X, \lambda]$ is needed, see (4.1). Similarly, we define quantities $\nu_p([X, \lambda], \mathbb{F}_q)$ and $\nu_\infty([X, \lambda], \mathbb{F}_q)$ using, respectively, equidistribution considerations for σ-conjugacy classes in $\mathrm{GSp}_{2g}(\mathbb{Q}_q)$ and the Sato-Tate measure on the compact form USp_{2g}.

Careless optimism might lead one to hope that $\widetilde{\#} I([X, \lambda], \mathbb{F}_q)$ is given by the product of the average archimedean and p-adic masses with the local deviations:

$$\widetilde{\#} I([X, \lambda], \mathbb{F}_q) \propto \nu_\infty([X, \lambda], \mathbb{F}_q) \prod_\ell \nu_\ell([X, \lambda], \mathbb{F}_q). \tag{1.2}$$

This argument is (at best) superficially plausible. Nonetheless, in this paper we give a pure-thought proof of the following theorem:

Theorem A. *Let $[X, \lambda]$ be a principally polarized abelian variety over $\mathbb{F}_q$ with commutative endomorphism ring. Suppose that either X is ordinary or that $\mathbb{F}_q = \mathbb{F}_p$ is the prime field. Then*

$$\widetilde{\#} I([X, \lambda], \mathbb{F}_q) = q^{\frac{\dim(\mathcal{A}_g)}{2}} \tau_T \nu_\infty([X, \lambda], \mathbb{F}_q) \prod_\ell \nu_\ell([X, \lambda], \mathbb{F}_q). \tag{1.3}$$

Here $\dim(\mathcal{A}_g) = \frac{g(g+1)}{2}$ *and* τ_T *is the Tamagawa number of the algebraic torus associated with* $[X, \lambda]$ *(defined below).*

As we have mentioned, this formulation is inspired by [Gek03], in which Gekeler proves Theorem A for an ordinary elliptic curve E over a finite prime field $\mathbb{F}_p$. (In the case $g = 1$ considered by Gekeler, τ_T equals 1.) Roughly speaking, the strategy there is to compute the terms ν_ℓ explicitly, and show that the right-hand side of (1.3) actually computes, via Euler products, the value at $s = 1$ of a suitable L-function. One concludes via the analytic class number formula and the known description of the isogeny class $I(E, \mathbb{F}_q)$ as a torsor under the class group of the quadratic imaginary order attached to the Frobenius of E. This strategy was redeployed in [AW15] and [GW19] for certain ordinary abelian varieties.

More recently, in [AG17], the first- and last-named authors showed directly that the right-hand side of (1.3) actually computes the product of the volume of a certain (adelic) quotient and an orbital integral on GL_2. Thanks to the work of Langlands [Lan73], and Dirichlet's class number formula, one has a direct proof that this product computes the size of the isogeny class of the elliptic curve.

In fact, this formula of Langlands, originally developed to count points on modular curves over finite fields, has been generalized by Kottwitz to an essentially arbitrary Shimura variety of PEL type [Kot92]. Kottwitz's formula (see Proposition 2.1 below), as in the case of Langlands, comes as a product of an (adelic) volume of a torus and an orbital integral, this time over GSp_{2g}. Let us remark that although the orbital integral in Kottwitz's and Langlands' formulas clearly decomposes as a product of local terms, the volume term, however, appears as a global quantity (a class number in the case of GL_2, cf. Lemma A.4 of [AG17]). Therefore an Euler product expression for $\widetilde{\#}I([X, \lambda], \mathbb{F}_q)$ such as the one in (1.3) is, at least, not immediate.

The content of the present paper is to prove that the Euler product given by the right-hand side of (1.3) is indeed equal to the product of the global volume and the orbital integral given by Kottwitz's formula. We establish this by incorporating the works of Shyr [Shy77] and by a delicate analysis of the interplay between various measures on the relevant spaces.

This paper is the logical extension of [AG17], which worked out these details for the case where the governing group is $\mathrm{GSp}_2 = \mathrm{GL}_2$. The reader will correctly expect that the structure of the argument is largely similar. However, the cohomological and combinatorial intricacies of symplectic similitude groups in comparison to general linear groups – in particular, the tori are much more complex and conjugacy and stable conjugacy need not coincide – mean that each stage is considerably more involved.

We highlight three particular issues that make the generalization from elliptic curves to higher rank not straight-forward.

The first is already mentioned above – the difference between conjugacy and stable conjugacy in GSp_{2g} when $g > 1$. This issue is discussed in detail in Section 3, and leads to the definition 4.1, which (as we prove in Section 3) coincides with (1.1) when $\ell \nmid p \operatorname{disc}(f)$.

The second is the fundamental lemma for base change, which is used to relate a Gekeler-style ratio at p to the twisted orbital integral. The complicated function one generally gets as a result of base change is the reason we have to assume that X is ordinary if $q \neq p$; this is discussed in detail in 4.3.

The last is that the tori in GSp_{2g} for $g \geq 2$ are significantly more complicated than those for $g = 1$. We handle the problems caused by the complexity of the tori using the work of Shyr [Shy77], which is the content of Section 5. Let us also remark that the global calculation in Section 5 involves

the Tamagawa number of the algebraic torus T. This number is well-known to be 1 for $g = 1$, but for general g we have to leave it as an (unknown) constant; Thomas Rüd and (independently) Wen-Wei Li obtained suggestive partial results and kindly agreed to present them in Appendix A.

Finally, we remark that (1.3), perhaps not surprisingly, can also be interpreted as a Smith-Minkowski-Siegel type mass formula (in the sense of Tamagawa-Weil) with explicit local masses (cf. [GY00]). Here the underlying group, of course, is GSp_{2g} and the masses calculate sizes of the relevant isogeny classes. Although this point of view is interesting in its own right we do not pursue it further in this paper. We would, however, like to note that the appearance of Tamagawa numbers is natural in this context.

Acknowledgment. JDA would like to thank Edgar Costa, Stefano Marseglia, David Roe and Christelle Vincent for sharing preliminary data on isogeny classes. SAA would like to thank Sungmun Cho and Jacob Tsimerman for various discussions. JG is grateful to William Casselman, Sug-Woo Shin, and Nicolas Templier for several helpful discussions, and to Cornell University, Association for Women in Mathematics, and the Michler family, as part of her work was carried out during a semester at Cornell University enabled by the Ruth Michler memorial prize. Our collaboration also benefited from NSERC funding. SAA's work is partially supported by the NSF (DMS1702176) and a startup grant from Boston University, JDA's work was partially supported by grants from the NSA (H98230-14-1-0161, H98230-15-1-0247 and H98230-16-1-0046).

Notation. We work over a finite field $\mathbb{F}_q = \mathbb{F}_{p^e}$ of characteristic p. We will often drop the field from all notation.

Fix a positive integer g. Let $V = \mathbb{Z}^{\oplus 2g}$, endowed with basis $x_1, \cdots, x_g, y_1, \cdots y_g$, and equip it with the symplectic form such that $\langle x_i, y_j \rangle = \delta_{ij}$ and $\langle x_i, x_j \rangle = \langle y_i, y_j \rangle = 0$. Let G be the group of similitudes $G = \mathrm{GSp}(V, \langle \cdot, \cdot \rangle) \cong \mathrm{GSp}_{2g}$; it has dimension $2g^2 + g + 1$ and rank $r = g + 1$. Its derived group is $G^{\mathrm{der}} \cong \mathrm{Sp}_{2g}$, and $G/G^{\mathrm{der}} \cong \mathbb{G}_m$. Let $\eta : G \to \mathbb{G}_m$ be the corresponding surjection (the multiplier map). We write T_{spl} for the split torus of diagonal matrices in G.

If K is a field, $\alpha \in G(K)$, and $\Gamma \subseteq G(K)$, we let ${}^{\Gamma}\alpha = \{\beta^{-1}\alpha\beta : \beta \in \Gamma\}$ be the orbit of α under Γ.

For an element $\gamma \in G(\mathbb{Q}_\ell)$, where ℓ is an arbitrary prime, and G is a split reductive algebraic group, the Weyl discriminant of γ is denoted by $D(\gamma)$: $D(\gamma) = \prod_{\alpha \in \Phi}(1 - \alpha(\gamma))$, where the product is over all roots of G (see §6.1.1 for details).

2. Background

2.1. **The Kottwitz formula.** The key formula we need is developed by Kottwitz in [Kot92]. In fact, the special case we need is detailed in [Kot90, Sec. 12]. By way of establishing necessary notation, we review the relevant part of this work here.

Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties of dimension g. An isogeny between two principally polarized abelian varieties $[X, \lambda], [Y, \mu] \in \mathcal{A}_g(\mathbb{F}_q)$ is an isogeny $\phi : X \to Y$ such that $m\phi^*\mu = n \cdot \lambda$ for some nonzero integers m and n. The isogeny class $I([X, \lambda], \mathbb{F}_q)$ is the set of all principally polarized abelian varieties $[Y, \mu]/\mathbb{F}_q$ admitting such an isogeny (over $\mathbb{F}_q$), and its weighted cardinality is

$$\widetilde{\#} I([X, \lambda], \mathbb{F}_q) = \sum_{[Y,\mu] \in I([X,\lambda],\mathbb{F}_q)} \frac{1}{\# \operatorname{Aut}(Y, \mu)}.$$

The abelian variety $X/\mathbb{F}_q$ admits a Frobenius endomorphism $\mathrm{Fr}_{X/\mathbb{F}_q}$, with characteristic polynomial $f_{X/\mathbb{F}_q}(T)$ of degree $2g$. (By [Tat66], this polynomial determines the isogeny class of X as an unpolarized abelian variety.)

For each $\ell \neq p$, $H^1(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell)$ (the dual of the Tate module) is a free $\mathbb{Z}_\ell$-module of rank $2g$, endowed with a symplectic pairing $\langle \cdot, \cdot \rangle_\lambda$ induced by the polarization. The Frobenius endomorphism $\varpi_{X/\mathbb{F}_q}$ induces an element $\gamma_{X/\mathbb{F}_q,\ell} \in \mathrm{GSp}(H^1(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell), \langle \cdot, \cdot \rangle_\lambda)$, and thus an element of $G(\mathbb{Z}_\ell)$, well-defined up to conjugacy. Moreover, there is an equality of characteristic polynomials $f_{\gamma_{X/\mathbb{F}_q,\ell}}(T) = f_{X/\mathbb{F}_q}(T)$. Simultaneously considering all finite primes $\ell \neq p$, we obtain an adelic similitude $\gamma_{[X,\lambda]} \in G(\mathbb{A}_f^p)$. (Alternatively one can, of course, directly consider the action of $\varpi_{X/\mathbb{F}_q}$ on $H^1(X_{\overline{\mathbb{F}}_q}, \hat{\mathbb{Z}}^p) = \lim_{\substack{\leftarrow \\ p \nmid n}} H^1(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/n)$.)

Similarly, the crystalline cohomology group $H^1_{\mathrm{cris}}(X, \mathbb{Q}_q)$ is endowed with an integral structure $H^1_{\mathrm{cris}}(X, \mathbb{Z}_q)$ and a σ-linear endomorphism F (where we denote by $\mathbb{Q}_q$ the degree e unramified extension of $\mathbb{Q}_p$, and by $\mathbb{Z}_q$ its ring of integers). It determines, up to σ-conjugacy, an element $\delta_{X/\mathbb{F}_q}$ of $G(\mathbb{Q}_q)$ with multiplier $\eta(\delta_{X/\mathbb{F}_q}) = p$.

The eth iterate of F is linear, and in fact F^e is the endomorphism of $H^1_{\mathrm{cris}}(X, \mathbb{Q}_q)$ induced by $\varpi_{X/\mathbb{F}_q}$.

Let $T_{[X,\lambda]}/\mathbb{Q}$ represent the automorphism group of $[X, \lambda]$ in the category of abelian varieties up to $\mathbb{Q}$-isogeny. Concretely, the polarization λ induces a (Rosati) involution $(\dagger)$ on $\mathrm{End}(X) \otimes \mathbb{Q}$; for each $\mathbb{Q}$-algebra R, we have

$$T_{[X,\lambda]}(R) = \{\alpha \in (\mathrm{End}(X) \otimes R)^\times : \alpha\alpha^{(\dagger)} \in R^\times\}.$$

By Tate's theorem [Tat66], for $\ell \neq p$, $T_{[X,\lambda]}(\mathbb{Q}_\ell) \cong G_{\gamma_{X/\mathbb{F}_q,\ell}}(\mathbb{Q}_\ell)$, the centralizer of $\gamma_{X/\mathbb{F}_q,\ell}$, and $T_{[X,\lambda]}(\mathbb{Q}_q) \cong G_{\delta_{X/\mathbb{F}_q}\sigma}(\mathbb{Q}_p)$, the twisted centralizer of $\delta_{X/\mathbb{F}_q}$ in $G(\mathbb{Q}_q)$.

A direct analysis of the effect of isogenies on the first cohomology groups of abelian varieties then shows:

Proposition 2.1 ([Kot90]). *The weighted cardinality of the isogeny class of $[X, \lambda] \in \mathcal{A}_g(\mathbb{F}_q)$ is*

$$\begin{aligned} \widetilde{\#} I([X,\lambda], \mathbb{F}_q) = \mathrm{vol}(T_{[X,\lambda]}(\mathbb{Q}) \backslash T_{[X,\lambda]}(\mathbb{A}_f)) \cdot \int_{G_{\gamma_{X/\mathbb{F}_q}}(\mathbb{A}_f^p) \backslash G(\mathbb{A}_f^p)} \mathbb{1}_{G(\hat{\mathbb{Z}}_f^p)}(g^{-1}\gamma_{X/\mathbb{F}_q} g)\, dg \\ \cdot \int_{G_{\delta_{X/\mathbb{F}_q}\sigma}(\mathbb{Q}_p) \backslash G(\mathbb{Q}_q)} \mathbb{1}_{G(\mathbb{Z}_q)\, \mathrm{diag}(p, \cdots, p, 1, \cdots, 1) G(\mathbb{Z}_q)}(h^{-1}\delta_{X/\mathbb{F}_q} h^\sigma)\, dh. \end{aligned} \tag{2.1}$$

In the orbital and twisted orbital integrals in (2.1), we choose the Haar measures on G which assign volume 1 to $G(\hat{\mathbb{Z}}^p)$ and to $G(\mathbb{Z}_q)$, respectively. The choice of measure on T does not matter here (as long as the same measure is used to calculate the global volume). We define the specific measure on T that we use below, in §5. It coincides with the canonical measure at all but finitely many places.

This formula appears in [Kot90, p.205]; see also [Kot92] for its generalization to a much larger class of PEL Shimura varieties. As in [AG17, 2.4], the weighted cardinality accounts for the fact that we have not introduced a rigidifying level structure, and thus our objects admit nontrivial, albeit finite, automorphism groups.

Remark 2.2. Using Honda–Tate theory, one can ([Kot90, p.206], [Kot92, p.422]) find $\gamma_{X/\mathbb{F}_q,0} \in G(\mathbb{Q})$, well-defined up to $G(\overline{\mathbb{Q}})$-conjugacy, such that $\gamma_{X/\mathbb{F}_q,0}$ and $\gamma_{X/\mathbb{F}_q,\ell}$ are conjugate in $G(\overline{\mathbb{Q}}_\ell)$. Similarly, $\gamma_{X/\mathbb{F}_q,0}$ and $\mathrm{N}\,\delta_{X/\mathbb{F}_q}$ are conjugate in $G(\overline{\mathbb{Q}}_q)$, where N denotes the norm map

$$\begin{aligned} G(\mathbb{Q}_q) &\longrightarrow G(\mathbb{Q}_p) \\ \alpha &\longmapsto \alpha\alpha^{\sigma}\cdots\alpha^{\sigma^{e-1}}. \end{aligned}$$

(The characteristic polynomial of $\gamma_{X/\mathbb{F}_q,0}$ is $f_{X/\mathbb{F}_q}(T)$.) Then the group variety $T_{[X,\lambda]/\mathbb{F}_q}$ is isomorphic to the centralizer of $\gamma_{X/\mathbb{F}_q,0}$ in G.

It turns out that moreover, one can find a rational element $\gamma_0 \in G(\mathbb{Q})$ such that γ_0 is $G(\mathbb{Q}_\ell)$-conjugate to $\gamma_{X/\mathbb{F}_q,\ell}$ for every $\ell \neq p$ (see [Kis17, p.889]). Consequently, in (2.1) we could replace $\gamma_{X/\mathbb{F}_q}$ with a global object γ_0; but we will never use this fact in this paper.

In the remainder of this paper we fix a principally polarized abelian variety $[X,\lambda]/\mathbb{F}_q$ with commutative endomorphism ring $\mathrm{End}(X)$. (For example, any simple, ordinary abelian variety necessarily has a commutative endomorphism ring [Wat69, Thm. 7.2].) By Tate's theorem, the commutativity of $\mathrm{End}(X)$ is equivalent to the condition that $T_{[X,\lambda]}$ is a maximal torus in G.

To ease notation slightly, we will write δ_0 and T for $\delta_{X/\mathbb{F}_q}$ and $T_{[X,\lambda]}$, respectively. If ℓ is a fixed, notationally suppressed prime, we will sometimes write γ_0 for $\gamma_{X/\mathbb{F}_q,\ell}$; by Remark 2.2, one may equally well let γ_0 be the image of some choice γ_0 in $G(\mathbb{Q})$ (though we will not be using it).

2.2. **Structure of the centralizer.** For future use, we record some information about the centralizer $T = T_{[X,\lambda]}$. Recall that X is a g-dimensional abelian variety with commutative endomorphism ring. Then T is a maximal torus in G, and $K := \mathrm{End}(X)^0 = \mathrm{End}(X) \otimes \mathbb{Q}$ is a CM-algebra of degree $2g$ over $\mathbb{Q}$. Then K is isomorphic to a direct sum $K \cong \oplus_{i=1}^t K_i$ of CM fields, and the Rosati involution on $\mathrm{End}(X)$ induces a positive involution $a \mapsto \overline{a}$ on K, which in turn restricts to complex conjugation on each component K_i. Let $K^+ \subset K$ be the subalgebra fixed by the positive involution. Then $K^+ \cong \oplus_{i=1}^t K_i^+$, where K_i^+ is the maximal totally real subfield of K_i, and $[K^+ : \mathbb{Q}] = g$.

In general, if L is a field and M/L is a finite étale algebra, let $\mathbf{R}_{M/L}$ be Weil's restriction of scalars functor. The norm map $\mathrm{N}_{M/L}$ induces a map of tori $\mathbf{R}_{M/L}\mathbb{G}_m \to \mathbb{G}_m$, and the norm one torus is the kernel of this map:

$$1 \longrightarrow \mathbf{R}^{(1)}_{M/L}\mathbb{G}_m \longrightarrow \mathbf{R}_{M/L}\mathbb{G}_m \xrightarrow{\mathrm{N}_{M/L}} \mathbb{G}_m \longrightarrow 1.$$

With these preparations we have

$$T^{\mathrm{der}} := T \cap G^{\mathrm{der}} \cong \mathbf{R}_{K^+/\mathbb{Q}}\mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m,$$

and T sits in the diagram

$$\begin{tikzcd} 1 \arrow[r] & T^{\mathrm{der}} \arrow[r] \arrow[d, "\sim"] & T \arrow[r] \arrow[d, hook] & \mathbb{G}_m \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \mathbf{R}_{K^+/\mathbb{Q}}\mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m \arrow[r] & \mathbf{R}_{K/\mathbb{Q}}\mathbb{G}_m \arrow[r, "\mathbf{R}_{K^+/\mathbb{Q}}\mathrm{N}_{K/K^+}"] & \mathbf{R}_{K^+/\mathbb{Q}}\mathbb{G}_m \arrow[r] & 1. \end{tikzcd} \tag{2.2}$$

On points, we have

$$\begin{aligned} T(\mathbb{Q}) &= \{a \in K^\times : a\bar{a} \in \mathbb{Q}^\times\} \\ &= \{(a_1, \cdots, a_t) \in \oplus K_i^\times : \exists c \in \mathbb{Q}^\times : a_i\bar{a}_i = c\} \\ T^{\mathrm{der}}(\mathbb{Q}) &= \{a \in K^\times : a\bar{a} = 1\} \\ &= \{(a_1, \cdots, a_t) \in \oplus K_i^\times : a_i\bar{a}_i = 1\}. \end{aligned}$$

Let $\widetilde{T} = T^{\mathrm{der}} \times \mathbb{G}_m$. In the sequel, it will be useful to have an explicit isogeny $\alpha : \widetilde{T} \to T$, as well as a complementary isogeny $\beta : T \to \widetilde{T}$ such that $\alpha \circ \beta$ is the squaring map. On points, these maps are given by

$$\begin{array}{ccccc} T & \xrightarrow{\beta} & \widetilde{T} & \xrightarrow{\alpha} & T \\ a & \longmapsto & (a\bar{a}^{-1}, a\bar{a}) & & \\ & & (b, c) & \longmapsto & bc. \end{array}$$

2.3. **The Steinberg quotient.** Recall that we have fixed a maximal split torus T_{spl} in G; let W be the Weyl group of G relative to T_{spl}. Let $T_{\mathrm{spl}}^{\mathrm{der}} = T_{\mathrm{spl}} \cap G^{\mathrm{der}}$, and let $A^{\mathrm{der}} = T_{\mathrm{spl}}^{\mathrm{der}}/W$ be the Steinberg quotient for the semisimple group G^{der}. It is isomorphic to the affine space of dimension $r - 1 = g$. We let $\mathbb{A}_G = A^{\mathrm{der}} \times \mathbb{G}_m$ be the analogue of the Steinberg quotient for the reductive group G, and define a map

$$G \xrightarrow{\quad \mathfrak{c} \quad} \mathbb{A}_G \tag{2.3}$$

$$\gamma \longmapsto (\mathrm{tr}(\gamma), \mathrm{tr}(\wedge^2\gamma), \cdots, \mathrm{tr}(\wedge^g\gamma), \eta(\gamma)).$$

Note that $\eta(\gamma) = \mathrm{tr}(\wedge^{g+1}(\gamma))/\,\mathrm{tr}(\gamma)$; and if $\gamma \in G^{\mathrm{der}} \subset G$, then $\mathfrak{c}(\gamma) = (\mathfrak{c}^{\mathrm{der}}(\gamma), 1)$, where $\mathfrak{c}^{\mathrm{der}}$ is the usual Steinberg map.

2.4. **Truncations.** Let ℓ be any finite prime (including $\ell = p$). Let $\pi_n = \pi_{\ell,n} : \mathbb{Z}_\ell \to \mathbb{Z}_\ell/\ell^n$ be the truncation map. For any $\mathbb{Z}_\ell$-scheme $\mathcal{X}$, we denote by $\pi_n^{\mathcal{X}}$ the corresponding map

$$\pi_n^{\mathcal{X}} : \mathcal{X}(\mathbb{Z}_\ell) \longrightarrow \mathcal{X}(\mathbb{Z}_\ell/\ell^n)$$

induced by π_n. Given $S_n \subset \mathcal{X}(\mathbb{Z}_\ell/\ell^n)$, we will often set

$$\widetilde{S}_n = \pi_n^{-1}(S_n).$$

The projection maps π_n^G extend to a somewhat larger set of similitudes. Let $M(\mathbb{Z}_\ell)$ be the set of symplectic similitudes which stabilize the lattice $V \otimes \mathbb{Z}_\ell$;

$$M(\mathbb{Z}_\ell) = \mathrm{GSp}(V \otimes \mathbb{Q}_\ell) \cap \mathrm{End}(V \otimes \mathbb{Z}_\ell) \cong \mathrm{GSp}_{2g}(\mathbb{Q}_\ell) \cap \mathrm{Mat}_{2g}(\mathbb{Z}_\ell).$$

Inside this set, for each $d \geq 0$ we identify a subset

$$M(\mathbb{Z}_\ell)_d = \{A \in M(\mathbb{Z}_\ell) : \mathrm{ord}_\ell \det(A) \leq d\}.$$

Finally, let us denote by $M(\mathbb{Z}_\ell/\ell^n)_d$ the set

$$M(\mathbb{Z}_\ell/\ell^n)_d = \{A \in M(\mathbb{Z}_\ell/\ell^n) : \mathrm{ord}_\ell \det(A) \leq d\}.$$

Note that $M(\mathbb{Z}_\ell)_0 = G(\mathbb{Z}_\ell)$, and in the last definition, the condition on the determinant is not vacuous even if $d \gg n$, because it rules out the matrices of determinant zero.

With a certain amount of abuse, we introduce the following notion of "$M(\mathbb{Z}_\ell)_d$-conjugacy":

Definition 2.3. *If $\gamma \in M(\mathbb{Z}_\ell)$, and in particular if $\gamma \in G(\mathbb{Z}_\ell)$, we will write $\gamma \sim_{M(\mathbb{Z}_\ell)_d} \gamma_0$ if there exists some $A \in M(\mathbb{Z}_\ell)_d$ such that $A\gamma = \gamma_0 A$.*

Similarly, if $\overline{\gamma} \in M(\mathbb{Z}_\ell/\ell^n)$, we write $\overline{\gamma} \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0$ if there exists some $\overline{A} \in M(\mathbb{Z}_\ell/\ell^n)_d$ such that $\overline{A}\gamma = \pi_n(\gamma_0)\overline{A}$.

When n is small relative to d, truncations of $M(\mathbb{Z}_\ell)_d$- conjugate elements might not be $M(\mathbb{Z}_\ell/\ell^n)_d$-conjugate (since, e.g., all the elements $A \in M(\mathbb{Z}_\ell)$ satisfying $A\gamma = \gamma_0 A$ might project to 0 mod ℓ^n). Of course, this does not happen when $n \gg d$. We also note that trivially, if $\gamma \sim_{M(\mathbb{Z}_\ell)_{d_0}} \gamma_0$ for some d_0, then $\gamma \sim_{M(\mathbb{Z}_\ell)_d} \gamma_0$ for all $d \geq d_0$. The analogous statement holds for $\overline{\gamma} \in G(\mathbb{Z}_\ell/\ell^n)$ as long as $n \gg d$.

2.5. **Measures and integrals.** As in [AG17], we need to explicitly work out the relationships between several different natural measures on the ℓ-adic points of varieties, especially groups and group orbits. The definitions introduced in [AG17, §3] (where a little more historical perspective is briefly reviewed) go through with minimal changes. We recall the relevant notation here.

Serre-Oesterlé measure: In [Ser81, §3], Serre observed that for a smooth p-adic submanifold Y of $\mathbb{Z}_p^m$ of dimension d, there is a limit $\lim_{n\to\infty} |Y_n| p^{-nd}$, where Y_n is the reduction of Y modulo p^n (in our notation, $Y_n = \pi_n(Y)$). Moreover, Serre pointed out that this limit can be understood as the *volume* of Y with respect to a certain measure, which is canonical. The definition of this measure for more general sets Y was elaborated on by Oesterlé [Oes82] and by Veys [Vey92]. We refer to this measure as the Serre-Oesterlé measure, and denote it by μ^{SO}.

Measures on groups: Once and for all, we fix the measure $|dx|_\ell$ on the affine line $\mathbb{A}^1_{\mathbb{Q}_\ell}$ to be the translation-invariant measure such that $\mathrm{vol}_{|dx|_\ell}(\mathbb{Z}_\ell) = 1$. Then there are two fundamentally different approaches to defining measure. The first is, for any smooth algebraic variety $\mathcal{X}$ over $\mathbb{Q}_\ell$ with a non-vanishing top degree differential form ω on it, one gets the associated measure $|d\omega|_\ell$ on $\mathcal{X}(\mathbb{Q}_\ell)$. In particular, for a reductive group G, there is a canonical differential form ω_G, defined in the greatest generality by Gross [Gro97]. This gives a canonical measure $|d\omega_G|_\ell$ on $G(\mathbb{Q}_\ell)$. When G is split over $\mathbb{Q}$, this measure has an alternative description using point-counting over the finite field (i.e., it coincides with Serre-Oesterlé measure μ_G^{SO} defined above):

$$\int_{G(\mathbb{Z}_\ell)} |d\omega_G|_\ell = \frac{\#G(\mathbb{Z}/\ell)}{\ell^{\dim(G)}}. \tag{2.4}$$

This observation is originally due to A. Weil [Wei82], and is actually built into his definition of integration on adeles. Weil's classical observation is precisely what makes this paper possible.

For groups, there is a second approach. Start with a Haar measure and normalize it so that some given maximal subgroup has volume 1. The choice of a "canonical" compact subgroup in this approach could lead to very interesting considerations (and is one of the main points of [Gro97]), but in our situation only two easy cases are needed. For $G(\mathbb{Q}_\ell)$, the relevant maximal subgroup is $G(\mathbb{Z}_\ell)$, and for $T_{[X,\lambda]}(\mathbb{Q}_\ell)$, it is the unique maximal compact subgroup (discussed below in §5.1). We denote such a Haar measure on $G(\mathbb{Q}_\ell)$ by μ_G^{can}, and on $T(\mathbb{Q}_\ell)$ by ν_T, following [Shy77].

Geometric measure on orbits: This is a measure constructed in [FLN10] on a fiber of the Steinberg map $\mathfrak{c} : G \to \mathbb{A}_G$. Let ω_G be a volume form on G, and let ω_A be the volume form $\bigwedge dx_i \wedge \frac{dx}{|x|}$ on $\mathbb{A}_G \cong \mathbb{A}^{\mathrm{rank}(G)-1} \times \mathbb{G}_m$. On the fiber $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))$, factor ω_G as

$$\omega_G = \omega^{\mathrm{geom}}_{\mathfrak{c}(\gamma)} \wedge \omega_A;$$

integrating $\left|\omega^{\mathrm{geom}}_{\mathfrak{c}(\gamma)}\right|$ defines a measure μ^{geom} on $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))$.

Suppose ϕ is a locally constant compactly supported function on $G(\mathbb{Q}_\ell)$. Recall the family $\gamma_{X/\mathbb{F}_q,\ell}$ (and δ_0), whose centralizers are the sets of $\mathbb{Q}_\ell$-points of the algebraic torus $T := T_{[X,\lambda]}$. We use two different measures on the orbit ${}^{G(\mathbb{Q}_\ell)}\gamma_{X/\mathbb{F}_q,\ell} \cong T_{[X,\lambda]}(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)$ to define an integral. When ℓ is fixed, we will often denote the element $\gamma_{X/\mathbb{F}_q,\ell}$ by γ_0; we let $\mu^{\mathrm{Shyr}}_{\gamma_0}$ be the quotient measure $\mu^{\mathrm{can}}_G/\nu_T$, and let $\mu^{\mathrm{geom}}_{\gamma_0}$ be the geometric measure reviewed above. (Since the orbit of γ_0 is an open subset of $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))$, the restriction of the geometric measure from $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))$ to the orbit makes sense.) Then for $\bullet \in \{\mathrm{Shyr}, \mathrm{can}, \mathrm{geom}\}$, set

$$O^\bullet_{\gamma_0} := \int_{T(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \phi(g^{-1}\gamma_0 g)\, d\,\mu^\bullet_{\gamma_0}.$$

3. Conjugacy

3.1. **Integral conjugacy.** To relate the right-hand side of (2.1) to the ratios ν_ℓ of (1.1), we interpret the orbital integral as the volume of the intersection of the $G(\mathbb{Q}_\ell)$-orbit of γ_0 with $G(\mathbb{Z}_\ell)$. For almost all ℓ, $G(\mathbb{Z}_\ell) \cap {}^{G(\mathbb{Q}_\ell)}\gamma_0 = {}^{G(\mathbb{Z}_\ell)}\gamma_0$:

Lemma 3.1. *Suppose $\gamma_0 \in G(\mathbb{Z}_\ell)$ and $\ell \nmid D(\gamma_0)$. If $\gamma \in G(\mathbb{Z}_\ell)$, then*

$$\gamma \sim_{G(\mathbb{Q}_\ell)} \gamma_0 \iff \gamma \sim_{G(\mathbb{Z}_\ell)} \gamma_0.$$

Proof. The hypothesis on γ_0 implies that the centralizer G_{γ_0} is a smooth torus over $\mathbb{Z}_\ell$, and thus the transporter from G_γ to G_{γ_0} is smooth over $\mathbb{Z}_\ell$ (e.g., [Con14, Prop. 2.1.2]).

Since γ and γ_0 are conjugate in $G(\mathbb{Q}_\ell)$, they have the same characteristic polynomial, and thus their reductions $\overline{\gamma}_0 = \pi^G_1(\gamma_0)$ and $\overline{\gamma} = \pi^G_1(\gamma)$ are stably conjugate in $G(\mathbb{Z}_\ell/\ell)$. By Lang's theorem, $\overline{\gamma}_0$ and $\overline{\gamma}$ are conjugate in $G(\mathbb{Z}_\ell/\ell)$; by smoothness of the transporter scheme, γ and γ_0 are conjugate in $G(\mathbb{Z}_\ell)$. □

If $\overline{\gamma}_0$ is *not* regular, then the set $G(\mathbb{Z}_\ell) \cap {}^{G(\mathbb{Q}_\ell)}\gamma_0$ generally consists of several different $G(\mathbb{Z}_\ell)$-orbits. Nonetheless, the number of distinct orbits is bounded; and membership in ${}^{G(\mathbb{Q}_\ell)}\gamma_0$ can be detected at a finite truncation level.

Lemma 3.2. *Suppose $\gamma_0 \in G(\mathbb{Z}_\ell)$ is regular semisimple. There exists an integer $e = e(\gamma_0)$ such that, if $n \gg 0$ and $d > e$, then for $\gamma \in G(\mathbb{Z}_\ell/\ell^n)$, the following conditions are equivalent:*

(1) $\gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0 \bmod \ell^n$, *and*
(2) *there exists some $\widetilde{\gamma} \in G(\mathbb{Z}_\ell)$ such that $\widetilde{\gamma} \bmod \ell^n = \gamma$ and $\widetilde{\gamma} \sim_{G(\mathbb{Q}_\ell)} \gamma_0$.*

The statement is also true with $G(\mathbb{Z}_\ell)$ replaced with $M(\mathbb{Z}_\ell)$ everywhere.

Proof. We prove the original statement.

The intersection of $G(\mathbb{Z}_\ell)$ with the $G(\mathbb{Q}_\ell)$-orbit of γ_0 is a finite union of $G(\mathbb{Z}_\ell)$-orbits, since it is compact (recall that γ_0 is regular semisimple) and the $G(\mathbb{Z}_\ell)$-orbits are open in this intersection; let $g_1, ..., g_s$ be representatives of these orbits, and let $A_i \in G(\mathbb{Q}_\ell)$ be elements satisfying $A_i g_i A_i^{-1} = \gamma_0$, so that $A_i g_i = \gamma_0 A_i$. We clear denominators; for each i, let $X_i \in M(\mathbb{Z}_\ell)$ be a scalar multiple of A_i. Then $X_i g_i = \gamma_0 X_i$, and we set

$$e(\gamma_0) = \max_{i \in \{1,..,s\}} \{|\mathrm{ord}(\det X_i)|\}.$$

Now, suppose $n > 2d(\gamma_0)$, where $d(\gamma_0)$ is the valuation of the discriminant of γ_0, $e \geq e(\gamma_0)$, and $n \gg e$. We want to prove that with these assumptions, an element $\gamma \in G(\mathbb{Z}_\ell/\ell^n)$ satisfies

$$\gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_e} \pi_n(\gamma_0)$$

if and only if there exists a lift $\widetilde{\gamma} \in G(\mathbb{Z}_\ell)$ such that $\pi_n(\widetilde{\gamma}) = \gamma$ and $\widetilde{\gamma} \sim_{G(\mathbb{Q}_\ell)} \gamma_0$.

One direction is easy: suppose there exists $\widetilde{\gamma} \in G(\mathbb{Z}_\ell)$ such that $\widetilde{\gamma} \bmod \ell^n = \gamma$ and $\widetilde{\gamma} \sim_{G(\mathbb{Q}_\ell)} \gamma_0$. Then there exists $i \in \{1, \ldots, s\}$ such that $\widetilde{\gamma} \sim_{G(\mathbb{Z}_\ell)} g_i$. Therefore there exists $Y \in G(\mathbb{Z}_\ell)$ such that $Y\widetilde{\gamma} = g_i Y$. Recall that as above, there exists $X_i \in M(\mathbb{Z}_\ell)$ such that $X_i g_i = \gamma_0 X_i$. Then $Z := \pi_n^M(X_i Y)$ lies in $M(\mathbb{Z}_\ell/\ell^n)_e$ and satisfies the condition $Z\gamma = \pi_n(\gamma_0)Z$.

The other direction is a special case of Hensel's Lemma. Since Hensel's Lemma in this generality, though well-known, is surprisingly hard to find in the literature, we provide a detailed explanation with references.

For each n, let

$$R_{\gamma_0}(\mathbb{Z}_\ell/\ell^n) = \{(A, \gamma) : A \in M(\mathbb{Z}_\ell/\ell^n), \gamma \in G(\mathbb{Z}_\ell/\ell^n), A\gamma = \pi_n^G(\gamma_0)A\} \subset M(\mathbb{Z}_\ell/\ell^n) \times G(\mathbb{Z}_\ell/\ell^n),$$

where π_n^G is the projection from §2.4. This is a system of $(2g)^2$ equations in $8g^2$ variables (namely, the matrix entries of A and γ). Now Hensel's Lemma as stated in [Bou85, III.4.5., Corollaire 3, p.271] applies directly, as follows. Let $n(\gamma_0)$ be the valuation of the minor formed by the first $(2g)^2$ columns of the Jacobian matrix of this system of equations at γ_0. By Hensel's lemma, if $n > 2n(\gamma_0)$ and $(A, \gamma) \in R_{\gamma_0}(\mathbb{Z}_\ell/\ell^n)$, then there exists some $\widetilde{\gamma} \in G(\mathbb{Z}_\ell)$ such that $\pi_n(\widetilde{\gamma}) = \gamma$ and $\widetilde{\gamma} \sim_{M(\mathbb{Z}_\ell)} \gamma_0$.

Since the core argument simply relies on the solvability, via Hensel's lemma, of a system of equations over $\mathbb{Z}_\ell$, it is also valid if $G(\mathbb{Z}_\ell)$ is replaced by $M(\mathbb{Z}_\ell)$. □

Remark 3.3. We observe (though we do not need this observation in this paper) that $n(\gamma_0)$ in fact equals the valuation of the discriminant of γ_0, e.g. by the argument provided in [Kot05, §7.2].

For $\gamma_0 \in G(\mathbb{Z}_\ell)$, let

$$C_{(d,n)}(\gamma_0) = \{\gamma \in G(\mathbb{Z}_\ell/\ell^n) : \gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0\}. \tag{3.1}$$

If $d = 0$, this coincides with the usual conjugacy class of $\pi_n(\gamma_0)$. As in Section 2.4, let $\widetilde{C}_{(d,n)}(\gamma_0) = (\pi_n^G)^{-1}(C_{(d,n)}(\gamma_0))$ be the set of lifts of elements of $C_{(d,n)}(\gamma_0)$ to $G(\mathbb{Z}_\ell)$.

We also extend this notation to elements $\gamma_0 \in M(\mathbb{Z}_\ell)$:

$$C_{(d,n)}(\gamma_0) = \{\gamma \in M(\mathbb{Z}_\ell/\ell^n) : \gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0\}.$$

(If $\gamma_0 \in G(\mathbb{Z}_\ell) \subset M(\mathbb{Z}_\ell)$, the two notions coincide and thus there is no ambiguity.)

Corollary 3.4. (a) *Suppose $\gamma_0 \in G(\mathbb{Z}_\ell)$. There exists $d = d(\gamma_0)$ such that, if $n \gg 0$, then*

$$C_{(d,n)}(\gamma_0) = \pi_n(G(\mathbb{Z}_\ell) \cap^{G(\mathbb{Q}_\ell)} \gamma_0).$$

Moreover,

$$G(\mathbb{Z}_\ell) \cap^{G(\mathbb{Q}_\ell)} \gamma_0 = \bigcap_{n \geq 0} \widetilde{C}_{(d,n)}(\gamma_0).$$

(b) *Suppose $\gamma_0 \in M(\mathbb{Z}_\ell)$. There exists $d = d(\gamma_0)$ such that, if $n \gg 0$, then*

$$C_{(d,n)}(\gamma_0) = \pi_n(M(\mathbb{Z}_\ell) \cap^{G(\mathbb{Q}_\ell)} \gamma_0).$$

Moreover,

$$M(\mathbb{Z}_\ell) \cap^{G(\mathbb{Q}_\ell)} \gamma_0 = \bigcap_{n \geq 0} \widetilde{C}_{(d,n)}(\gamma_0).$$

Proof. This is a direct consequence of Lemma 3.2. □

3.2. **Stable (twisted) conjugacy.** In this section, we further assume that $[X, \lambda]$ is a principally-polarized abelian variety with commutative endomorphism ring for which $1/2$ is not a slope of the Newton polygon of X. (Again, any ordinary simple principally polarized abelian variety satisfies these hypotheses.)

Recall the definition of K and K^+, as well as the discussion of T, from Section 2.2. By a prime of K (or K^+) lying over p we mean a prime $\mathfrak{p}$ of some K_i (respectively, K_i^+) lying over p, and we write $K_\mathfrak{p}$ for $K_{i,\mathfrak{p}}$. With this convention, we then have $K \otimes \mathbb{Q}_p \cong \oplus K_\mathfrak{p}$.

Lemma 3.5. *Let $\mathfrak{p}^+$ be a prime of K^+ lying over p. Then $\mathfrak{p}^+$ splits in K.*

Proof. This is standard. We work in the category of p-divisible groups up to isogeny. Then $X[p^\infty]$ has height $2g$, and comes equipped with an action by $K \otimes_\mathbb{Q} \mathbb{Q}_p$.

Corresponding to the decomposition $K^+ \otimes_\mathbb{Q} \mathbb{Q}_p \cong \oplus_{\mathfrak{p}^+|p} K^+_{\mathfrak{p}^+}$ we have the decomposition

$$X[p^\infty] = \oplus X[\mathfrak{p}^{+\infty}].$$

Moreover, $X[\mathfrak{p}^{+\infty}]$ is a p-divisible group of height $2[K^+_{\mathfrak{p}^+} : \mathbb{Q}_p]$, and self-dual (because $\mathfrak{p}^+\mathcal{O}_K$ is stable under the Rosati involution). We now fix one $\mathfrak{p}^+$, and show that it must split in K.

Since $K^+_{\mathfrak{p}^+}$ is a field (and not just a $\mathbb{Q}_p$-algebra) of dimension $\frac{1}{2}\operatorname{ht}(X[\mathfrak{p}^{+\infty}])$, $X[\mathfrak{p}^{+\infty}]$ has at most two slopes. Since by hypothesis $1/2$ is not a slope of X, $X[\mathfrak{p}^{+\infty}]$ has exactly two slopes, say $\lambda = a/b$ and $1 - a/b$, where $\gcd(a, b) = 1$. Let m be the multiplicity of λ as a slope of $X[\mathfrak{p}^{+\infty}]$; then $mb = [K^+_{\mathfrak{p}^+} : \mathbb{Q}_p]$. The endomorphism algebra of $X[\mathfrak{p}^{+\infty}]$ (again, in the category of p-divisible groups up to isogeny) is isomorphic to

$$\operatorname{End}(X[\mathfrak{p}^{+\infty}])^0 \cong \operatorname{Mat}_m(D_\lambda) \oplus \operatorname{Mat}_m(D_{1-\lambda}),$$

where D_λ is the central simple $\mathbb{Q}_p$-algebra with Brauer invariant λ. In particular, any subfield L of $\operatorname{End}(X[\mathfrak{p}^{+\infty}])^0$ satisfies $[L : \mathbb{Q}_p] \leq mb = [K^+_{\mathfrak{p}^+} : \mathbb{Q}_p]$. Since $K \otimes_{K^+} K^+_{\mathfrak{p}^+}$ acts on $X[\mathfrak{p}^{+\infty}]$, we must have $K \otimes_{K^+} K^+_{\mathfrak{p}^+} \cong K^+_{\mathfrak{p}^+} \oplus K^+_{\mathfrak{p}^+}$, as claimed. □

Corollary 3.6. *We have*

$$T^{\mathrm{der}}_{\mathbb{Q}_p} \cong \oplus_{\mathfrak{p}^+} \mathbf{R}_{K^+_{\mathfrak{p}^+}/\mathbb{Q}_p} \mathbb{G}_m.$$

Proof. Since $T^{\mathrm{der}} = \mathbf{R}_{K^+/\mathbb{Q}}\mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m$, using Lemma 3.5 we find

$$\begin{aligned} T^{\mathrm{der}}_{\mathbb{Q}_p} &= \mathbf{R}_{K^+\otimes\mathbb{Q}_p/\mathbb{Q}_p}\mathbf{R}^{(1)}_{K\otimes\mathbb{Q}_p/K^+\otimes\mathbb{Q}_p}\mathbb{G}_m \\ &= \cong \oplus_{\mathfrak{p}^+|p}\, \mathbf{R}_{K^+_{\mathfrak{p}^+}/\mathbb{Q}_p}\mathbf{R}^{(1)}_{K\otimes_{K^+}K^+_{\mathfrak{p}^+}/K^+_{\mathfrak{p}^+}}\mathbb{G}_m. \end{aligned}$$

If L is any field then $\mathbf{R}_{L\oplus L/L}\mathbb{G}_m \cong \mathbb{G}_{m,L} \oplus \mathbb{G}_{m,L}$; the norm map $\mathbf{R}_{L\oplus L/L}\mathbb{G}_m \to \mathbb{G}_m$ is given by multiplication of components; and so $\mathbf{R}^{(1)}_{L\oplus L/L}\mathbb{G}_m$ is isomorphic to $\mathbb{G}_{m,L}$, where the latter is embedded in the former via $(\mathrm{id}, \mathrm{inv})$. □

Recall that we have chosen $\delta_0 \in G(\mathbb{Q}_q)$ (well-defined up to σ-conjuacy) and $\gamma_0 = \mathrm{N}\,\delta_0 \in G(\mathbb{Q}_p)$.

Lemma 3.7. *The stable conjugacy class of γ_0 consists of a single conjugacy class, and the stable σ-conjugacy class of δ_0 consists of a single σ-conjugacy class.*

Proof. To prove the first claim, it suffices (by [Kot82, p.788]) to show that $H^1(\mathbb{Q}_p, T)$ vanishes. By taking the long exact sequence of cohomology of the top row of (2.2), and then invoking Hilbert 90 and Corollary 3.6, we find that $H^1(\mathbb{Q}_p, T)$ does in fact vanish.

For the second claim, it similarly suffices to show that the first cohomology of the twisted centralizer $G_{\delta_0,\sigma}$ vanishes [Kot82, p.805]. However, the twisted centralizer of an element is always an inner form of the (usual) centralizer of its norm [Kot82, Lemma 5.8]. In our case, the centralizer $T = G_{\gamma_0}$ is a torus, and thus admits no nontrivial inner forms. We conclude again that $H^1(\mathbb{Q}_p, G_{\delta_0,\sigma})$ is trivial. □

4. Ratios

4.1. **Definitions.** For $\ell \neq p$, we define a local ratio $\nu_\ell([X,\lambda])$ designed to measure the extent to which the conjugacy class of $\gamma_{X_0/\mathbb{F}_q}$, as an element of $G(\mathbb{Z}_\ell/\ell)$, is more or less prevalent than a randomly chosen conjugacy class. More precisely, to measure this probability, we consider the finite group $G(\mathbb{Z}_\ell/\ell^n)$ for sufficiently large n, and recall that our notion of "conjugacy" in this group is not the usual conjugacy but the relation $\sim_{M(\mathbb{Z}_\ell/\ell^n)_e}$ defined above in §2.4. For $\ell = p$, the element $\gamma_{X_0/\mathbb{F}_q}$ is not in $G(\mathbb{Z}_p)$, and we use $M(\mathbb{Z}_p)$ instead; but this has no effect on the definition since our notion of "conjugacy" in $G(\mathbb{Z}_p/p^n)$ already uses $M(\mathbb{Z}_p)$.

Recall the definition of $C_{(d,n)}(\gamma_0)$ from (3.1), and that $C_n(\gamma_0) := C_{(0,n)}(\gamma_0)$ is the actual conjugacy class of $\pi_n(\gamma_0)$ in $G(\mathbb{Z}_\ell/\ell^n)$.

Definition 4.1. *For each finite place ℓ, including $\ell = p$, using the shorthand $\gamma_0 := \gamma_{X/\mathbb{F}_q,\ell} \in M(\mathbb{Z}_\ell)$, set*

$$\nu_\ell([X,\lambda]) = \nu_\ell([X,\lambda],\mathbb{F}_q) = \lim_{d\to\infty}\lim_{n\to\infty}\frac{\#C_{(d,n)}(\gamma_0)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)}. \tag{4.1}$$

At infinity, define

$$\nu_\infty([X,\lambda]) = \nu_\infty([X,\lambda],\mathbb{F}_q) = \frac{|D(\gamma_0)|_\infty^{1/2}}{(2\pi)^g} \tag{4.2}$$

where $|\cdot|_\infty$ is the usual real absolute value.

Remark 4.2. So far we have avoided using the fact that there exists a rational element $\gamma_0 \in G(\mathbb{Q})$ as in Remark 2.2, and treated γ_0 as an element of $G(\mathbb{A}_f)$. We can continue doing so, and then for (4.2) simply *define* the archimedean absolute vaue of its discriminant by $|D(\gamma_0)|_\infty := \prod_\ell \left|D(\gamma_{X/\mathbb{F}_q,\ell})\right|_\ell^{-1}$.

The ratios stabilize for large enough d and n, and thus the limits are not, strictly speaking, necessary. In fact, for $\ell \neq 2, p$ and not dividing the discriminant of γ_0, the ratios stabilize right away, at $d = 0$ and $n = 1$, as the next two lemmas show.

Lemma 4.3. *If $\ell \nmid D(\gamma_0)$ and $n \gg 0$, then $C_{(d,n)}(\gamma_0) = C_n(\gamma_0)$.*

Proof. Recall that all our notation assumes that ℓ is fixed. Clearly $C_n(\gamma_0) \subseteq C_{(d,n)}(\gamma_0)$. Conversely, suppose n is sufficiently large as in Lemma 3.2 and that $\gamma \in C_{(d,n)}(\gamma_0)$. Then there exists some $\widetilde{\gamma} \in \pi_n^{-1}(\gamma)$ such that $\widetilde{\gamma} \sim_{G(\mathbb{Q}_\ell)} \gamma_0$. By Lemma 3.1, $\widetilde{\gamma} \sim_{G(\mathbb{Z}_\ell)} \gamma_0$, and so $\gamma \sim_{G(\mathbb{Z}_\ell/\ell^n)} \gamma_0$. □

Lemma 4.4. *If $\ell \nmid D(\gamma_0)$, $\ell \neq 2$, and $d \gg_{\gamma_0} 0$, then for $n \geq 1$,*

$$\frac{\#C_{(d,n)}(\gamma_0)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)} = \frac{\#\{\gamma \in G(\mathbb{Z}_\ell/\ell) : \gamma \sim \gamma_0\}}{\#G(\mathbb{Z}_\ell/\ell)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell)}. \tag{4.3}$$

Proof. By Lemma 4.3, the left-hand side of (4.3) is

$$\frac{\#G(\mathbb{Z}_\ell/\ell^n)/\#G_{\gamma_0}(\mathbb{Z}_\ell/\ell^n)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)} = \frac{\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)}{\#G_{\gamma_0}(\mathbb{Z}_\ell/\ell^n)}.$$

Since $\pi_1(\gamma_0)$ is regular, the centralizer G_{γ_0} is smooth over $\mathbb{Z}_\ell$ of relative dimension $g+1$. Since the same is true of the scheme $\mathbb{A}_G/\mathbb{Z}_\ell$, the result now follows. □

4.2. **From ratios to integrals.** Fix a prime ℓ (possibly $\ell = p$ or $\ell = 2$). (In this subsection, as above, all quantities depend on this notationally suppressed prime.) Recall (2.3) the canonical map $\mathfrak{c} : G \to A$ from G to its Steinberg quotient. The fibres of this map over regular points are stable orbits of regular semisimple elements. Define a system of neighbourhoods of $\mathfrak{c}(\gamma_0)$ inside $\mathbb{A}_G(\mathbb{Z}_\ell)$ by

$$\widetilde{U}_n(\gamma_0) = \widetilde{\pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma_0))} = (\pi_n^{\mathbb{A}_G})^{-1}(\pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma_0))).$$

In other words,

$$\widetilde{U}_n(\gamma_0) = \{a = (a_1, \dots a_g, \eta) \in \mathbb{A}_G(\mathbb{Z}_\ell) \mid a_i \equiv a_i(\gamma_0) \bmod \ell^n, \eta \equiv \eta(\gamma_0) \bmod \ell^n\}.$$

These definitions and (3.1) are summarized by the diagram

$$\begin{array}{ccc} \widetilde{C}_{(d,n)}(\gamma_0) \subset M(\mathbb{Z}_\ell) & \xrightarrow{\mathfrak{c}} & \mathbb{A}_G(\mathbb{Z}_\ell) \supset \widetilde{U}_n(\gamma_0) \\ \downarrow \pi_n^M & & \downarrow \pi_n^{\mathbb{A}_G} \\ C_{(d,n)}(\gamma_0) \subset M(\mathbb{Z}_\ell/\ell^n) & \xrightarrow{\mathfrak{c}_n} & \mathbb{A}_G(\mathbb{Z}/\ell^n) \ni \pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma_0)), \end{array} \tag{4.4}$$

where $\mathfrak{c}_n : G(\mathbb{Z}/\ell^n) \to \mathbb{A}_G(\mathbb{Z}/\ell^n)$ is the map sending an element to the coefficients of its characteristic polynomial mod ℓ^n. The diagram of maps commutes since reduction mod ℓ^n is a ring homomorphism, and the map $\mathfrak{c}$ is polynomial in the matrix entries of γ. (The diagram of subsets need not commute, though.) We also note that when $\ell \neq p$, the sets $\widetilde{C}_{(d,n)}(\gamma_0)$ and $C_{(d,n)}(\gamma_0)$ are contained in $G(\mathbb{Z}_\ell)$ and $G(\mathbb{Z}_\ell/\ell^n)$, respectively, since $\mathrm{ord}_\ell(\det(\gamma_0)) = 0$ in this case, and this is also true for all elements that are congruent to any conjugate of γ_0.

By definition of the geometric measure, for any open subset $B \subset G(\mathbb{Z}_\ell)$ we have

$$\mathrm{vol}_{\mu^{\mathrm{geom}}}(B \cap \mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))) = \lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_G|}(\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap B)}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}. \tag{4.5}$$

Recall that each stable orbit $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))$ is a finite disjoint union of rational orbits. Each rational orbit being an open subset of the stable orbit, we may and do define geometric measure on each rational orbit, by restriction.

In simple terms, the sets $\widetilde{U}_n$ form a system of neighbourhoods of the point $\mathfrak{c}(\gamma_0) \in \mathbb{A}_G$; the set $\widetilde{C}_{(d,n)}(\gamma_0)$ can be thought of as the intersection of a neighbourhood of the orbit of γ_0 with $G(\mathbb{Z}_\ell)$; the set $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))$ is the stable orbit of γ_0. The following lemma gives the precise relationships between all these sets.

Lemma 4.5. (a) *Let $\ell \neq p$. For large enough d and n (depending on γ_0), we have*

$$\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap G(\mathbb{Z}_\ell) = \bigcup_{\gamma' \sim_{G(\overline{\mathbb{Q}}_\ell)} \gamma_0} \widetilde{C}_{d,n}(\gamma'), \tag{4.6}$$

where γ' runs over a set of representatives of $G(\mathbb{Q}_\ell)$-conjugacy classes in the stable conjugacy class of γ_0 whose $\mathbb{Q}_\ell$-orbits intersect $G(\mathbb{Z}_\ell)$, so that we may take the elements γ' to lie in $G(\mathbb{Z}_\ell)$.

(b) *When n is sufficiently large (depending on γ_0), the sets $\widetilde{C}_{d,n}(\gamma')$ above are disjoint.*

(c) *Let μ_G^{SO} be the Serre-Oesterlé measure on $G(\mathbb{Q}_p) \cap M(\mathbb{Z}_p)$, viewed as a submanifold of $M(\mathbb{Z}_p)$. Then $\mathrm{vol}_{\mu_G^{\mathrm{SO}}}(\widetilde{C}_{d,n}(\gamma_0)) = \ell^{-n\dim(G)}\#C_{d,n}(\gamma_0)$; in particular, if $\ell \neq p$, $\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{d,n}(\gamma_0)) = \ell^{-n\dim(G)}\#C_{d,n}(\gamma_0)$.*

Proof. **(a).** This is an easy consequence of the fact that two regular semisimple elements of $G(\mathbb{Q}_\ell)$ are stably conjugate if and only if their characteristic polynomials coincide. In our notation,

$$\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0)) = \sqcup_{\gamma' \sim_{G(\overline{\mathbb{Q}_\ell})} \gamma_0} ({}^{G(\mathbb{Q}_\ell)}\gamma'),$$

where ${}^{G(\mathbb{Q}_\ell)}\gamma'$ denotes the rational conjugacy class of γ' in $G(\mathbb{Q}_\ell)$ as before. Now, we will describe both the left-hand side and the right-hand side of (4.6) as: the set of elements $\gamma \in G(\mathbb{Z}_\ell)$ whose characteristic polynomial is congruent to that of γ_0 mod ℓ^n. Indeed, on the left-hand side, by definition, $\gamma \in \mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap G(\mathbb{Z}_\ell)$ if and only if $\pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma)) = \pi_n^{\mathbb{A}_G}(\mathfrak{c}(\gamma_0))$. By the commutativity of (4.4), this is equivalent to $\mathfrak{c}_n(\pi_n^G(\gamma)) = \mathfrak{c}_n(\pi_n^G(\gamma_0))$, i.e., the characteristic polynomials of γ and γ_0 are congruent modℓ^n. On the right-hand side, given $\gamma' \in G(\mathbb{Z}_\ell)$, by Lemma 3.2, for d and n large enough [1], we have that $\gamma \in \widetilde{C}_{(d,n)}(\gamma')$ if and only if there exists $\gamma'' \in G(\mathbb{Z}_\ell)$ such that $\gamma'' \equiv \gamma'$ mod ℓ^n and γ'' is $G(\mathbb{Q}_\ell)$-conjugate to γ. Taking the union of these sets as γ' runs over the set of integral representatives of $G(\mathbb{Q}_\ell)$-conjugacy classes in the stable class of γ_0, we obtain the set of all elements $\gamma \in G(\mathbb{Z}_\ell)$ that are congruent modulo ℓ^n to an element of $G(\mathbb{Z}_\ell)$ that is stably conjugate to γ_0, i.e., to an element having the same characteristic polynomial as γ_0. This means that $\mathfrak{c}_n(\pi_n^G(\gamma)) = \mathfrak{c}_n(\pi_n^G(\gamma_0))$, which completes the proof of the first statement.

(b). Since the orbits of regular semisimple elements are closed in the ℓ-adic topology, distinct orbits have disjoint neighbourhoods.

(c). The map $\pi_n^M : \widetilde{C}_{(d,n)}(\gamma_0) \to C_{(d,n)}(\gamma_0)$ is surjective, so $\widetilde{C}_{(d,n)}(\gamma_0)$ can be thought of as a disjoint union of fibres of π_n^M. Since M is a smooth scheme over $\mathbb{Z}_\ell$, each fibre of π_n^M has volume $\ell^{-n\dim(G)}$ with respect to the measure μ^{SO} (cf. [Ser81]). The first statement follows. Moreover, as discussed above in §2.5, on $G(\mathbb{Z}_\ell)$, the measures μ^{SO} and $\mu_{|\omega_G|}$ coincide. For $\ell \neq p$, we have $\widetilde{C}_{(d,n)}(\gamma_0) \subset G(\mathbb{Z}_\ell)$, which completes the proof. □

Recall that ϕ_0 is the characteristic function of $G(\mathbb{Z}_\ell)$.

[1] *Large enough* depends on γ', but only through its discriminant. Since stably conjugate elements have the same discriminant, ultimately this only depends on γ_0.

Corollary 4.6. *Let $\ell \neq p$. Then there exists $d(\gamma_0)$ such that for $d \geq d(\gamma_0)$*

$$O^{\mathrm{geom}}_{\gamma_0}(\phi_0) = \lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{(d,n)}(\gamma_0))}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}.$$

Proof. The orbital integral, by definition, calculates the volume of the set of integral points in the rational orbit of γ_0, with respect to the geometric measure on the orbit. Using Lemma 4.5(a)-(b) we write $\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap G(\mathbb{Z}_\ell) = \sqcup_{\gamma'} \widetilde{C}_{d,n}(\gamma')$, where γ' are as in that lemma, with γ_0 being one of the elements γ'. The union on the right-hand side of (4.6) is a disjoint union of neighbourhoods of the individual orbits, intersected with $G(\mathbb{Z}_\ell)$. The statement follows from the equality (4.5), applied to the set $B := \widetilde{C}_{d,n}(\gamma_0)$. □

Corollary 4.7. *For $\ell \neq p$, the Gekeler ratio* (4.1) *is related to the geometric orbital integral by*

$$\nu_\ell([X,\lambda]) = \frac{\ell^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell)} O^{\mathrm{geom}}_{\gamma_0}(\phi_0).$$

Proof. Note that at a finite level n (and for d large enough so that the equalities in all the previous lemmas hold), the denominator in (4.1) is

$$\frac{\#G(\mathbb{Z}_\ell/\ell^n)}{\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)} = \frac{\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell^n)\#\mathbb{G}_m(\mathbb{Z}/\ell^n)}{\ell^{(\mathrm{rank}(G)-1)n}\#\mathbb{G}_m(\mathbb{Z}/\ell^n)} = \frac{\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell^n)}{\ell^{(\mathrm{rank}(G)-1)n}} = \frac{\ell^{(\dim(G)-1)(n-1)}\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell)}{\ell^{(\mathrm{rank}(G)-1)n}}.$$

By Lemma 4.5(c), we have $\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{d,n}(\gamma_0)) = \#C_{d,n}(\gamma_0)/\ell^{n\dim(G)}$, and by definition of the measure on the Steinberg quotient, $\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0)) = \ell^{-n\,\mathrm{rank}(G)}$ (here we are using the fact that $|\eta(\gamma_{X,\ell})| = 1$ for $\ell \neq p$, so the absolute value of the $\mathbb{G}_m$-coordinate is 1 on $\widetilde{U}_n$).

Then for a given level n, we have

$$\begin{aligned}\frac{\#C_{(d,n)}(\gamma_0)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)} &= \frac{\ell^{n\dim(G)}\,\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{(d,n)}(\gamma_0))\ell^{(\mathrm{rank}(G)-1)n}}{\ell^{(\dim(G)-1)(n-1)}\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell)} \\ &= \frac{\ell^{\dim(G)-1}}{\#G^{\mathrm{der}}(\mathbb{Z}_\ell/\ell)} \frac{\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{(d,n)}(\gamma_0))}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}.\end{aligned}$$

The result now follows from Corollary 4.6. □

4.3. **Calculation at p.** Recall that we have fixed a maximal split torus $T_{\mathrm{spl}} \subset G$. For any cocharacter $\lambda \in X_*(T_{\mathrm{spl}})$ (and any power $q = p^e$ of p), let $\psi_\lambda = \psi_{\lambda,q}$ be the characteristic function of the double coset

$$D_{\lambda,q} = G(\mathbb{Z}_q)\lambda(p)G(\mathbb{Z}_q).$$

By the Cartan decomposition, the collection of all ψ_λ is a basis for $\mathcal{H}_G = \mathcal{H}_{G,\mathbb{Q}_q}$, the Hecke algebra of functions on $G(\mathbb{Q}_q)$ which are bi-$G(\mathbb{Z}_q)$-invariant.

Let μ_0 be the cocharacter $p \mapsto \mathrm{diag}(p,\cdots,p,1,\cdots,1)$; it is the cocharacter associated to the Shimura variety $\mathcal{A}_g$. Define

$$\begin{aligned}\psi_{q,p} &= \psi_{\mu_0,q} = \mathbb{1}_{G(\mathbb{Z}_q)\,\mathrm{diag}(p,\cdots,p,1,\cdots,1)G(\mathbb{Z}_q)} \\ \phi_{q,p} &= \psi_{e\mu_0,p} = \mathbb{1}_{G(\mathbb{Z}_p)\,\mathrm{diag}(q,\cdots,q,1,\cdots,1)G(\mathbb{Z}_p)}.\end{aligned}$$

Recall that $\delta_0 = \delta_{X/\mathbb{F}_q}$ represents the absolute Frobenius of X, and that $\gamma_0 = \mathrm{N}\,\delta_0$.

Lemma 4.8. *Let $[X, \lambda]/\mathbb{F}_q$ be a principally polarized abelian variety. Suppose that either X is ordinary or that $q = p$ (and thus $e = 1$). Then*

$$^{G(\mathbb{Q}_p)}\gamma_0 \cap M(\mathbb{Z}_p) \subseteq D_{e\mu_0,p}.$$

Proof. We identify $X_*(T_{\mathrm{spl}})$ with

$$\{\underline{a} = (a_1, \cdots, a_{2g}) \in \mathbb{Z}^{2g} : a_i + a_{g+i} = a_j + a_{g+j} \text{ for } 1 \leq i, j \leq g\}. \tag{4.7}$$

Suppose $\gamma \in D_{\underline{a},p} \subseteq G(\mathbb{Q}_p)$. Then $\mathrm{ord}_p(\eta(\gamma))$ is the common value of $a_i + a_{g+i}$; and γ stabilizes $V \otimes \mathbb{Z}_p$ – that is, $\gamma \in M(\mathbb{Z}_p)$ – if and only if each $a_i \geq 0$.

Let $f(\underline{a}) = \#\{i : a_i = 0\}$. If $\alpha \in D_{\underline{a}} \cap M(\mathbb{Z}_p)$, then $f(\underline{a})$ is the rank of $\pi_1(\alpha)$ as an endomorphism of V/pV.

With these preparations, suppose $\gamma \in^{G(\mathbb{Q}_p)} \gamma_0 \cap M(\mathbb{Z}_p)$. Note that we have $a_i + a_{g+i} = e$.

First, suppose X is ordinary. Then exactly g eigenvalues of γ are p-adic units. Consequently, if $\gamma \in D_{\underline{a}}$, then $f(\underline{a}) = g$. The only $\underline{a}$ as in (4.7) compatible with the symmetry and integrality constraints is $(e, \cdots, e, 0, \cdots, 0)$.

Second, suppose X has arbitrary Newton polygon but that $e = 1$. Again, the only $\underline{a}$ such that $a_i + a_{g+i} = e = 1$ and each $a_i \geq 0$ is $(1, \cdots, 1, 0, \cdots, 0)$. □

Lemma 4.9. *Suppose that $[X, \lambda]/\mathbb{F}_q$ is an ordinary, simple, principally polarized abelian variety. Then*

$$TO_{\delta_0}(\psi_{q,p}) = O_{\gamma_0}(\phi_{q,p}).$$

Proof. There is a base change map $b = b_{G,\mathbb{Q}_q/\mathbb{Q}_p} : \mathcal{H}_{G,\mathbb{Q}_q} \to \mathcal{H}_{G,\mathbb{Q}_p}$. The fundamental lemma asserts that, if $\psi \in \mathcal{H}_{G,\mathbb{Q}_q}$, then stable twisted orbital integrals for ψ match with stable orbital integrals for $b\psi$. For our δ_0 and γ_0, the adjective *stable* is redundant (Lemma 3.7), the case of the fundamental lemma we need is [Clo90, Thm. 1.1], and we have

$$TO_{\delta_0}(\psi_{q,p}) = O_{\gamma_0}(b\psi_{q,p}). \tag{4.8}$$

While we will stop short of computing $b\psi_{q,p}$, we will find a function which agrees with it on the orbit $^{G(\mathbb{Q}_p)}\gamma_0$.

The Satake transformation is an algebra homomorphism $\mathfrak{S} : \mathcal{H}_{G,\mathbb{Q}_q} \to \mathcal{H}_{T_{\mathrm{spl}},\mathbb{Q}_q}$ which maps $\mathcal{H}_{G,\mathbb{Q}_q}$ isomorphically onto the subring $\mathcal{H}^W_{T_{\mathrm{spl}},\mathbb{Q}_q}$ of invariants under the Weyl group. It is compatible with base change, in the sense that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_{G,\mathbb{Q}_q} & \xrightarrow{\mathfrak{S}} & \mathcal{H}_{T_{\mathrm{spl}},\mathbb{Q}_q} \cong \mathbb{C}[X_*(T_{\mathrm{spl}})] \\ \downarrow b & & \downarrow b \\ \mathcal{H}_{G,\mathbb{Q}_p} & \xrightarrow{\mathfrak{S}} & \mathcal{H}_{T_{\mathrm{spl}},\mathbb{Q}_p} \end{array}$$

We exploit the following data about the Satake transform and the base change map.

Under the canonical identification of $X_*(T_{\mathrm{spl}})$ and $X^*(\hat{T}_{\mathrm{spl}})$, the character group of the dual torus, $\lambda \in X_*(T_{\mathrm{spl}})$ gives rise to a character of $\hat{T}_{\mathrm{spl}}$, and thus a representation V_λ of $\hat{G}$; let χ_λ be its trace. We have

$$\mathfrak{S}(\psi_{\mu,q}) = q^{\langle \mu,\rho \rangle}\chi_\mu + \sum_{\lambda<\mu} a(\mu, \lambda)\chi_\lambda$$

for certain numbers $a(\mu, \lambda)$, where as usual ρ is the half-sum of positive roots [Gro98, (3.9)].

On one hand, following Gross [Gro98, (3.15)] and Kottwitz [Kot84b, (2.1.3)], we observe that the weight $\mu_0 = (1, \dots 1, 0, \dots, 0)$ is miniscule, and therefore

$$\mathfrak{S}(\psi_{\mu_0,q}) = q^{\langle \mu_0, \rho \rangle} \chi_{\mu_0}.$$

If we think of elements of $\mathbb{C}[X_*(T_{\mathrm{spl}})]^W$ as polynomials in $2g$ variables $z_1, \dots, z_{2g}$, then (essentially by definition of the highest weight and the fact that the multiplicity of the highest weight in an irreducible representation is 1 – in our case the representation in question is in fact the oscillator representation [Gro98, (3.15)]) we find that the leading term of $\mathfrak{S}(\psi_{\mu_0,q})$ is $q^{\langle \mu_0, \rho \rangle} z_1 \dots z_g$. By definition, the base change map takes $f \in \mathbb{C}[z_1, \dots, z_{2g}, z_1^{-1}, \dots, z_{2g}^{-1}]^W$ to $f(z_1^e, \dots, z_{2g}^e)$. Then

$$b(\mathfrak{S}(\psi_{q,p})) = q^{\langle \mu_0, \rho \rangle} z_1^e \dots z_g^e + \sum_{\lambda < e\mu_0} a(e\mu_0, \lambda) \chi_\lambda.$$

On the other hand, we have

$$\begin{aligned} \mathfrak{S}(\phi_{q,p}) &= p^{\langle e\mu_0, \rho \rangle} \chi_{e\mu_0} + \sum_{\lambda < e\mu_0} b(e\mu_0, \lambda) \chi_\lambda \\ &= q^{\langle \mu_0, \rho \rangle} z_1^e \dots z_g^e + \sum_{\lambda < e\mu_0} c(e\mu_0, \lambda) \chi_\lambda. \end{aligned}$$

In these formulas, $a(e\mu_0, \lambda)$, $b(e\mu_0, \lambda)$ and $c(e\mu_0, \lambda)$ are coefficients of lower weight monomials that are ultimately irrelevant to our calculation. In particular,

$$\phi_{q,p} - \mathfrak{S}^{-1}(b(\mathfrak{S}(\psi_{q,p})))$$

vanishes on $D_{e\mu_0,p} = G(\mathbb{Z}_p) e\mu_0(p) G(\mathbb{Z}_p)$.

The last point to note is that the intersection of the support of this difference $\phi_{q,p} - \mathfrak{S}^{-1}(b(\mathfrak{S}(\psi_{q,p})))$ with the orbit of γ_0 is contained in $M(\mathbb{Z}_p)$. Once we have shown this, the desired result follows from the fundamental lemma (4.8) combined with Lemma 4.8. We start by observing that since the multiplier is a multiplicative map, it is constant on double $G(\mathbb{Z}_p)$-cosets. Therefore, for any double coset $D_{\underline{a},p}$ such that $D_{\underline{a},p} \cap {}^{G(\mathbb{Q}_p)}\gamma_0 \neq \emptyset$, we have $a_i + a_{g+i} = e$. Now suppose $\lambda \leftrightarrow \underline{a}$ is a dominant weight satisfying this condition and further satisfying $\lambda \leq e\mu_0$. Then we have $a_1 \geq a_2 \geq \dots \geq a_g$ and $a_g \geq 0$ because λ is dominant; and on the other hand, $e - a_1 \geq e - a_2 \geq \dots \geq e - a_g$, and $e - a_g \geq 0$ because of the condition $\lambda \leq e\mu_0$. Therefore in particular, $a_{g+1}, \dots, a_{2g}$ are non-negative, and thus $D_{\lambda,p} \subset M(\mathbb{Z}_p)$ (and in fact, we have also shown that $a_1 = \dots = a_g$). □

Lemma 4.10. *Suppose that either X is ordinary or that $q = p$. Then there exists $d(\gamma_0)$ such that*

$$O_{\gamma_0}^{\mathrm{geom}}(\phi_{q,p}) = \lim_{n \to \infty} \frac{\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{d(\gamma_0),n}(\gamma_0))}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}.$$

Proof. Suppose that X is ordinary (but q is an arbitrary power of p). By Lemma 3.7, $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma_0))$ is a single $G(\mathbb{Q}_p)$-conjugacy class; the same argument shows this is true for elements in a small neighbourhood of $\mathfrak{c}(\gamma_0)$. Thus, using (4.5), $O_{\gamma_0}^{\mathrm{geom}}(\phi_{q,p})$ equals $\lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_G|}\left(\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap D_{e\mu_0,p}\right)}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}$. By Lemma 4.8, we have

$$\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap D_{e\mu_0,p} = \mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap M(\mathbb{Z}_p).$$

Therefore, all we need to show is that for large enough d and n, we have

$$\mathfrak{c}^{-1}(\widetilde{U}_n(\gamma_0)) \cap M(\mathbb{Z}_p) = \widetilde{C}_{d,n}(\gamma_0); \tag{4.9}$$

but this is essentially Corollary 3.4(b).

The case where $q = p$ follows from Lemma 4.6 and the second case of Lemma 4.8. □

Lemma 4.11. *On the double coset $D_{e\mu_0,p}$ we have*

$$|d\omega_G| = q^{\frac{g(g+1)}{2}+1}\mu^{\mathrm{SO}}. \tag{4.10}$$

Proof. Let $K = G(\mathbb{Z}_p)$. First, observe that the measure μ^{SO} on $G(\mathbb{Q}_p) \cap M(\mathbb{Z}_p)$ is both left- and right- K-invariant (since multiplication by an element of $G(\mathbb{Z}_p)$ yields a bijection on $\bmod p^n$-points). Consider the decomposition of $D_{e\mu_0,p}$ into, say, left K-cosets: $D_{e\mu_0,p} = \sqcup_{i=1}^{s} g_i K$ (the number s of these cosets was computed by Iwahori and Matsumoto but is not needed here). It follows from left K-invariance of μ^{SO} that $\mu^{\mathrm{SO}}(g_i K)$ is the same for all i.

Second, the measure $|d\omega_G|$ is normalized so that each K-coset has volume $\#G(\mathbb{F}_p)$. Thus, in order to compare the measures μ^{SO} and $|d\omega_G|$, we need to compare the cardinality $\#\pi_n(g_i K)$ of the reduction $\bmod p^n$ of any such coset $g_i K$ that is contained in $D_{e\mu_0,p}$ with $\#G(\mathbb{F}_p)$, for sufficiently large n. (Note that $n = 1$ is insufficient, because for all such cosets the reduction $\bmod p$ of any matrix in gK would be of lower rank. One needs to go to $n > e$ for the ratios $\frac{\#\pi_n(gK)}{p^{n\dim(G)}}$ to stabilize.) Since the answer does not depend on g_i, we can take $g_0 = e\mu_0(p) = \mathrm{diag}(q, \dots, q, 1 \dots, 1)$. In other words, we need to compute the cardinality of the fibre of the map

$$\begin{aligned} \varphi_q : G(\mathbb{Z}/p^n) &\longrightarrow M(\mathbb{Z}/p^n) \\ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] &\longmapsto \left[\begin{smallmatrix} qA & qB \\ C & D \end{smallmatrix}\right]. \end{aligned}$$

For simplicity, we would like to move the calculation to the Lie algebra. Let $n \gg e$. Observe that if $\varphi_q(\gamma_1) = \varphi_q(\gamma_2)$ for $\gamma_1, \gamma_2 \in G(\mathbb{Z}/p^n)$, then $\left[\begin{smallmatrix} qI_g & 0 \\ 0 & I_g \end{smallmatrix}\right](\gamma_1\gamma_2^{-1} - I) = 0$, where I_g is the $g \times g$-identity matrix, and I is the identity matrix in M_{2g}. This implies, in particular, that $\gamma_1\gamma_2^{-1} \equiv I \bmod p^{n-e}$. Then we can write the truncated exponential approximation: $\gamma_1\gamma_2^{-1} = I + X + \frac{1}{2}X^2 + \dots$ for some $X \in \mathfrak{g}(\mathbb{Z}_p)$; in particular, there exists $X \in \mathfrak{g}(\mathbb{Z}_p)$ such that $\gamma_1\gamma_2^{-1} \equiv I + X \bmod p^{2(n-e)}$, and thus the kernel of the map φ_q is in bijection with the set of $(X \bmod p^n)$ for $X \in \mathfrak{g}(\mathbb{Z}_p)$ such that $\left[\begin{smallmatrix} qI_g & 0 \\ 0 & I_g \end{smallmatrix}\right] X \equiv 0 \bmod p^n$.

We have $\mathfrak{g} = \mathfrak{sp}_{2g} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the 1-dimensional Lie algebra of the centre. It will be convenient to decompose it further: let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sp}_{2g}$ consisting of diagonal matrices, and let V consist of matrices whose diagonal entries are all zero; then

$$\mathfrak{g} = (\mathfrak{z} \oplus \mathfrak{h}) \oplus V.$$

Consider the action of multiplication by $\left[\begin{smallmatrix} qI_g & 0 \\ 0 & I_g \end{smallmatrix}\right]$ on each term of this direct sum decomposition.

On the term $\mathfrak{z} \oplus \mathfrak{h}$ it acts by $\mathrm{diag}(a_1, \dots, a_{2g}) \mapsto \mathrm{diag}(qa_1, \dots, qa_g, a_{g+1}, \dots, a_{2g})$, which in the $\mathfrak{z} \oplus \mathfrak{h}$-coordinates can be written as (recalling that $a_i + a_{g+i} = z$ is independent of i):

$$\begin{aligned} &\frac{z}{2} \oplus \left(\frac{z}{2} - a_{g+1}, \dots \frac{z}{2} - a_{2g}, -\frac{z}{2} + a_{g+1}, \dots - \frac{z}{2} + a_{2g}\right) \\ &\mapsto \frac{qz}{2} \oplus \left(\frac{qz}{2} - \frac{(q+1)a_{g+1}}{2}, \dots \frac{qz}{2} - \frac{(q+1)a_{2g}}{2}, -\frac{qz}{2} + \frac{(q+1)a_{g+1}}{2}, \dots - \frac{qz}{2} + \frac{(q+1)a_{2g}}{2}\right). \end{aligned}$$

The only points $(z, a_{g+1}, \dots, a_{2g})$ that are killed $(\bmod p^n)$ by this map are of the form $(z', 0, \dots, 0)$ with $qz' = 0$; so there are q of them.

Next consider an element $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in V$. Then A is determined by D, and B is skew-symmetric (up to a permutation of rows and columns). Multiplication by $\left[\begin{smallmatrix} qI_g & 0 \\ 0 & I_g \end{smallmatrix}\right]$ scales each entry of A and B by a factor of q, and does not change C and D. Since A is determined by D, the elements X

killed by this map are in bijection with symmetric matrices B with entries in $\mathbb{Z}/p^n$ that are killed by multiplication by q. Since the space of such matrices is a $g(g+1)/2$-dimensional linear space, the number of such matrices B is $q^{g(g+1)/2}$.

Thus, we have computed that $|d\omega_G| = q^{\frac{g(g+1)}{2}+1}\mu^{\mathrm{SO}}$ on the double coset $D_{e\mu_0,p}$. Combining this with (4.11), we get:

$$\nu_p([X,\lambda]) = \frac{qp^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_p/p)} \frac{q^{-\frac{g(g+1)}{2}-1}\operatorname{vol}_{|d\omega_G|}(\widetilde{C}_{d,n}(\gamma_0))}{\operatorname{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))} = q^{-\frac{g(g+1)}{2}} \frac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_p/p)} O^{\mathrm{geom}}_{\gamma_0}(\phi_{q,p}),$$

which completes the proof. □

Corollary 4.12. *Suppose that either X is ordinary or that $q = p$. For $\ell = p$, the Gekeler ratio* (4.1) *is related to the geometric orbital integral by*

$$\nu_p([X,\lambda]) = q^{-\frac{g(g+1)}{2}} \frac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{Z}_p/p)} O^{\mathrm{geom}}_{\gamma_0}(\phi_{q,p}).$$

Proof. First observe that $\operatorname{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0)) = qp^{-n\operatorname{rank}(G)}$, since we are using the invariant measure on the $\mathbb{G}_m$-factor of $\mathbb{A}_G = \mathbb{A}^{\operatorname{rank}(G)-1} \times \mathbb{G}_m$, and for γ_0 (and therefore, for all points in $\widetilde{U}_n$), that coordinate is the multiplier, with absolute value q^{-1}. Thus, by Lemma 4.5 (c) and the same argument as in Corollary 4.7, we have that for $d > d(\gamma_0)$,

$$\nu_p([X,\lambda]) = \lim_{n\to\infty} \frac{\#C_{d,n}(\gamma_0)}{\#G(\mathbb{Z}/p^n\mathbb{Z})/\#\mathbb{A}_G(\mathbb{Z}/p^n\mathbb{Z})} = \frac{qp^{\dim(G)-1}}{\#G^{\mathrm{der}}(\mathbb{Z}/p\mathbb{Z})} \lim_{n\to\infty} \frac{\operatorname{vol}_{\mu^{\mathrm{SO}}}(\widetilde{C}_{d,n}(\gamma_0))}{\operatorname{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}. \tag{4.11}$$

The ratio inside the limit on the right-hand side is the same as the ratio in Lemma 4.10, except that the measure in the numerator is the Serre-Oesterlé measure μ^{SO} rather than the measure $|d\omega_G|$. (Both measures are defined on $G(\mathbb{Q}_p) \cap M(\mathbb{Z}_p)$.) Thus, to prove the corollary, we just need to compute the conversion factor between the restrictions of the measures μ^{SO} and $|d\omega_G|$ to the support of $\phi_{q,p}$, which is the content of Lemma 4.11. □

5. The product formula

Now that the relationship between the ratios ν_ℓ and orbital integrals (with respect to the geometric measure) is established, we can translate the formula of Langlands and Kottwitz (2.1) into a Siegel-style product formula for the ratios, thus obtaining our main theorem. Recall the notation of §2, in particular, the element $\gamma_{[X,\lambda]} \in G(\mathbb{A}_f)$ associated with the isogeny class of $[X,\lambda]$, and its centralizer $T = T_{[X,\lambda]}$. Here in order to ease the notation we drop all the subscripts $[X,\lambda]$. Note that there is some flexibility in the choice of the measures in the Langlands and Kottwitz formula, but the measures need to be normalized by normalizing the measures on $G(\mathbb{Q}_\ell)$ and on $T(\mathbb{Q}_\ell)$ separately, and both need to assign volume 1 to the maximal compact subgroups at almost all primes. We will use the canonical measure $d\mu_G^{\mathrm{can}}$ on $G(\mathbb{Q}_\ell)$ for every prime ℓ, and the measure ν_T (defined in detail below) on $T(\mathbb{Q}_\ell)$ for all ℓ. Since Gekeler-style ratios are expressed in terms of the geometric measure on orbits, we need to calculate the conversion factor between the geometric measure and the quotient $d\mu_G^{\mathrm{can}}/d\nu_T$. We prove (the definitions of the relevant invariants, such as the Tamagawa number, are reviewed below):

Theorem 5.1. *Let $\gamma = \gamma_{(X,\lambda)} \in \mathrm{GSp}_{2g}(\mathbb{A}_f)$, and let T be the centralizer of γ, as above. Let $T^{\mathrm{der}} = T \cap G^{\mathrm{der}} = T \cap \mathrm{Sp}_{2g}$. Let τ_T denote the Tamagawa number of the torus T. Then*

$$\mathrm{vol}_{\nu_T}(T(\mathbb{Q})\backslash T(\mathbb{A}^f)) \prod_\ell d\mu_\ell^{\mathrm{Shyr}} = |\eta(\gamma)|_\infty^{-\frac{g(g+1)}{4}} \frac{|D(\gamma)|_\infty^{1/2}\tau_T}{(2\pi)^g} \prod_\ell \left(\frac{\#\mathrm{Sp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})}{\ell^{\dim(\mathrm{Sp}_{2g})}} \right)^{-1} d\mu_\ell^{\mathrm{geom}}, \tag{5.1}$$

where μ_ℓ^{Shyr} is the quotient measure on the orbit of γ obtained as a quotient of the canonical measure on G by the measure that we denote by ν_T on T, which gives volume 1 to the maximal compact subgroup of $T(\mathbb{Q}_\ell)$ at each place.

(Here we recall the convention that as we are thinking of γ as an element of $G(\mathbb{A}^f)$, we denote by $|D(\gamma)|_\infty$ (respectively, $|\eta(\gamma)|_\infty$) the inverse of the product of the corresponding ℓ-adic absolute values.)

Once this theorem is proved, our main theorem follows easily. The proof of Theorem 5.1 will occupy almost all of this section.

Remark 5.2. The factor appearing on the right-hand side of (5.1) has a nice expression in terms of zeta-values:

$$\frac{\#\mathrm{Sp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})}{\ell^{\dim(\mathrm{Sp}_{2g})}} = \prod_{i=1}^{g} \zeta_\ell^{-1}(2i).$$

See Appendix A for a discussion of the Tamagawa number τ_T.

We start by reviewing the formula due to J.-M. Shyr, and the relevant definitions and notation.

5.1. **Shyr's formula.** Shyr's formula is a generalization of the analytic class number formula to algebraic tori. We will be applying it to T^{der} in the notation above, so we will especially pay attention to the case of a torus anisotropic at the archimedean prime.

Theorem 5.3. *(*[Shy77, Theorem 1]*) For S an algebraic torus defined over $\mathbb{Q}$,*

$$h_S \cdot R_S = \rho_S \cdot w_S \cdot \tau_S \cdot |D_S|^{1/2}. \tag{5.2}$$

Here we review the definitions of invariants involved in Shyr's result.

5.1.1. *Points of tori.* Following Shyr, we introduce the following (abstract) groups attached to the torus S.

The character group of S is the free $\mathbb{Z}$-module $X^*(S) = \hat{S}$ (we emphasize that this is the lattice of the characters defined over $\overline{\mathbb{Q}}$). Apart from using $X^*(S)$ for the module denoted $\hat{S}$ by Shyr, in this section we follow Shyr's notation even though it differs from the notation in modern use, since we need to quote his results in fine detail. In particular, if F is any field containing $\mathbb{Q}$, we let $(\hat{S})_F$ be the subgroup of characters of S which are defined over F. If v is any place of $\mathbb{Q}$, let $(\hat{S})_v = (\hat{S})_{\mathbb{Q}_v}$.

As usual, we have $S(\mathbb{A})$ and $S(\mathbb{A}_f)$, the points of S with values in, respectively, the ring of adeles and the ring of finite adeles. The (finite) adeles come equipped with the product absolute value $|\cdot|_{\mathbb{A}}$, and we set

$$S^1(\mathbb{A}) = \{s \in S(\mathbb{A}) : \forall \chi \in (\hat{S})_{\mathbb{Q}}, |\chi(s)|_{\mathbb{A}} = 1\}.$$

Similarly, for each place v of $\mathbb{Q}$, let

$$S_v^c = \{s \in S(\mathbb{Q}_v) : \forall \chi \in (\hat{S})_v, |\chi(s)| = 1\}.$$

This is the unique maximal compact subgroup of $S(\mathbb{Q}_v)$; if S happens to be anisotropic over $\mathbb{Q}_v$, then this condition is empty and $S_v^c = S(\mathbb{Q}_v)$.

We further define

$$\mathbf{S}^c = \mathbf{S}^c_{\mathbb{A}_f} = \prod_{v \text{ finite}} S_v^c$$
$$\mathbf{S}^c_{\mathbb{A}} = \prod_v S_v^c$$

and set

$$S(\infty) = S(\mathbb{R}) \times \mathbf{S}^c$$
$$S_{\mathbb{Q}}(\infty) = S(\mathbb{Q}) \cap S(\infty).$$

5.1.2. *Constants and measures.* We can now define all terms which arise in (5.2).

- **Quasi-residue ρ_S:** Let F be a Galois extension which splits S. Then the character lattice $X^*(S)$ can be viewed as a $\mathrm{Gal}(F/\mathbb{Q})$-module, and this module uniquely determines S up to isomorphism. We denote this representation by σ_S, and let $L(s, \sigma_S) = \prod_p L_p(s, \sigma_S)$ be the corresponding Artin L-function (see [Bit11] for a modern treatment). Let r be the multiplicity of the trivial representation in σ_S. By definition,
$$\rho_S := \lim_{s \to 1} (s-1)^r L(s, \sigma_S).$$
- **Invariant form ω_S:** Let ω be an invariant gauge form on S. (In particular, ω is defined over $\mathbb{Q}$.) Set
$$\omega_S = \omega_\infty \prod_v L_v(1, \sigma_{S,v}) \omega_p,$$
where ω_v is the invariant measure on $S(\mathbb{Q}_v)$ induced by ω.

  By the product formula, as long as ω is defined over $\mathbb{Q}$, none of the global invariants depend on the normalization of ω. We thus may and do assume that ω_∞ is the form dt/t on each copy of $\mathbb{R}^\times$ in $S_{/\mathbb{R}}$, and is the arc length on every copy of the unit circle in $S_{/\mathbb{R}}$.
- **Ratio D_S:** This number is defined using the ratio of two invariant measures (defined below) on $S(\mathbb{A})$:
$$D_S^2 = \frac{\nu_S}{\mu_S}.$$
- **Measure μ.** Let $\chi_1, \cdots, \chi_r$ be a basis for $(\hat{S})_{\mathbb{Q}}$, and define a map Λ by
$$\begin{aligned} S(\mathbb{A}) &\xrightarrow{\quad\Lambda\quad} (\mathbb{R}_+^\times)^r \\ x &\longmapsto (|\chi_1(x)|_{\mathbb{A}}, \cdots, |\chi_r(x)|_{\mathbb{A}}). \end{aligned}$$
(In the cases of interest, when $S = T^{\mathrm{der}}$ or $S = T$, we have $r = 0$ or $r = 1$, respectively.) Then Λ induces an isomorphism
$$\tilde{\Lambda} : S(\mathbb{A})/S^1(\mathbb{A}) \xrightarrow{\sim} (\mathbb{R}_+^\times)^r .$$
(Of course, both sides are trivial if S is anisotropic.)

  Define $d\tilde{t}$ by
$$d\tilde{t} := \tilde{\Lambda}^*(\prod_{k=1}^r \frac{dt}{t}).$$

Let $dS_{\mathbb{Q}}$ be the counting measure on $S(\mathbb{Q})$ and normalize the Haar measure $dS^1_{\mathbb{A}}$ on $S^1(\mathbb{A})$ to give total measure 1. Then

$$d\mu = d\tilde{t}(dS^1_{\mathbb{A}} \times dS_{\mathbb{Q}}).$$

- **Measure ν_S:** This measure is defined in a similar fashion, as follows. Let $\xi_1, \cdots, \xi_{r_v}$ be a $\mathbb{Z}$-basis for $(\hat{S})_v$, and define Λ_v by

$$\begin{aligned} S(\mathbb{Q}_v) &\xrightarrow{\quad\Lambda_v\quad} (\mathbb{R}^\times_+)^{r_v} \\ x &\longmapsto (|\xi_1(x)|_v, \cdots, |\xi_r(x)|_v). \end{aligned}$$

We have

$$\tilde{\Lambda}_v : S_v/S_v^c \simeq R_v := \begin{cases} (\mathbb{R}^\times_+)^{r_\infty} & v = \infty \\ \mathbb{Z}^{r_v} & v \neq \infty \end{cases}.$$

(Note that that $r_\infty = 0$ if S is anisotropic over $\mathbb{R}$.)

If $v = \infty$, endow R_v with the product of the measure $\frac{dt}{t}$ on each component; if v is finite, endow R_v with the counting measure. In each case, let dt_v be the pullback, via $\tilde{\Lambda}_v^*$, of this measure to S_v/S_v^c.

Finally, endow S_v^c with the Haar measure dS_v^c of total volume one, and set

$$\nu_S := \prod_v \nu_{S,v} = \prod_v (dS_v^c \times dt_v).$$

Then by definition,

$$\nu_T(\mathbf{S}^c) = \nu_S(\prod_{v \text{ finite}} S_v^c) = 1.$$

For $S = T$, this is the measure that we will use in the Langlands-Kottwitz formula.

- **Tamagawa number τ_S:** The Tamagawa number is defined by

$$\tau_S = \int_{S^1(\mathbb{A})/S(\mathbb{Q})} dm,$$

where

$$d\tilde{t}(dm \times dS_{\mathbb{Q}}) = \rho_S^{-1} d\omega_S.$$

- **Class number h_S:** The class number of S is the (finite) cardinality

$$h_S := \#S(\mathbb{Q})\backslash S(\mathbb{A})/S(\infty).$$

- **Global units w_S:** The analogue of the number of roots of unity in a restriction of scalars torus is

$$w_S = \#(\mathbf{S}^c_{\mathbb{A}} \cap S(\mathbb{Q})).$$

- **Regulator R_S:** We will not need this definition since for an anisotropic torus S, $R_S = 1$. (See [Shy77, p.370] for the definition).

Remark 5.4. Note that the definition of canonical measure gives volume 1 to the connected component of the Néron model $\mathcal{S}^\circ$ of S_v, for a finite place v. When v is unramified, the maximal compact subgroup S_v^c coincides with $\mathcal{S}^\circ(\mathbb{Z}_v)$. However, when v is ramified, $\mathcal{S}^\circ(\mathbb{Z}_v)$ is a subgroup of finite index in S_v^c. This relation is explored in detail in [Bit11], but here we avoid it by simply using the measure ν_S rather than the canonical measure on the torus.

We quote for future use a crucial, albeit almost tautological, identity for tori which are anisotropic at infinity.

Lemma 5.5. *(*[Shy77, Lemma 2]*) Let S be an algebraic torus defined over $\mathbb{Q}$ such that $S(\mathbb{R})$ is anisotropic. Then*

(a) $|D_S|^{-1/2} = \omega_\infty(S(\mathbb{R}))\prod_\ell L_\ell(1,\sigma_S)\,\mathrm{vol}_{\omega_S}(S_\ell^c)$, *and*
(b) $\omega_\infty(S(\mathbb{R})) = (2\pi)^{\dim S}$ *and* $R_S = 1$.

Here, of course, $\dim S$ is the dimension of this torus as an algebraic group.

Proof. **(a).** This is precisely [Shy77, Lemma 2] (from which we can read that the constant **c** of that Lemma is $|D_S|^{-1/2}$), with the only difference that the first factor on the right-hand side in his Lemma 2 is $\Phi_0^{-1}(I)$ (in his notation). Here I is the lattice defined in the first line of the proof of *loc. cit.* Its rank equals r_∞, so it is zero when $S(\mathbb{R})$ is compact. In this case the map $\Phi_0 : S(\mathbb{R}) \to \mathbb{R}_+^{r_\infty}$ is necessarily trivial, and thus $\Phi_0^{-1}(I) = S(\mathbb{R})$.

(b). Since $r = r_\infty = 0$ when S is anisotropic over $\mathbb{R}$, we have $R_S = 1$. (More precisely, in the proof of Lemma 2 of [Shy77], R_S is expressed as the ratio $\nu_S(\Phi^{-1}(P))/\nu_S(\Phi^{-1}(I))$; but both parallelotopes P and I are trivial in our case, since $r = r_\infty = 0$.)

By definition, ω_∞ is the gauge form on S. By the classification of real algebraic tori, the only possibility for $S(\mathbb{R})$ when S is anisotropic over $\mathbb{R}$ is $S(\mathbb{R}) = S^1 \times \cdots \times S^1$, the product of $\dim S$ circles. Then its volume with respect to the natural gauge form is $(2\pi)^{\dim S}$.

□

5.2. **The global volume term.** We have the exact sequence

$$1 \longrightarrow \mathbf{S}^c/\mathbf{S}^c \cap S(\mathbb{Q}) \longrightarrow S(\mathbb{A}_f)/S(\mathbb{Q}) \longrightarrow S(\mathbb{A})/S(\mathbb{Q})S(\infty) \longrightarrow 1.$$

Note that

$$\#\mathbf{S}^c \cap S(\mathbb{Q}) = w_S,$$

see e.g. [Shy77, Lemma 1].

Continuing with the calculation, since the measure of $\mathbf{S}^c$ with respect to ν_S is 1, we get

$$\mathrm{vol}_{\nu_S}(S(\mathbb{Q})\backslash S(\mathbb{A}_f)) = \frac{h_S}{w_S}. \tag{5.3}$$

5.3. **Passing to the derived subgroup.** Let $G = \mathrm{GSp}_{2g}$ and $\gamma \in \mathrm{GSp}_{2g}(\mathbb{A}^f)$ be such that the centralizer T of γ is an algebraic torus as in §2.2, defined over $\mathbb{Q}$, and in particular, $T^{\mathrm{der}} := T \cap \mathrm{Sp}_{2g}$ is anisotropic over $\mathbb{Q}$. Let μ_G^{can} and ν_T be the measures that give measure 1 to the maximal compacts of $G(\mathbb{Q}_\ell)$ and $T(\mathbb{Q}_\ell)$ at finite places. Let μ^{Shyr} denote the quotient measure $\mu_G^{\mathrm{can}}/\nu_T$ on the orbit of γ. Finally, let μ^{geom} denote the geometric measure as above.

We recall some more notation and results from Shyr's article. For a homomorphism $\theta : G \to H$ of abstract groups with finite kernel and cokernel, the symbol $q(\theta)$ stands for $|\mathrm{Coker}(\theta)|/|\ker(\theta)|$.

Recall that in §2.2 we have defined $\widetilde{T} = T^{\mathrm{der}} \times \mathbb{G}_m$ and explicit isogenies $\alpha : \widetilde{T} \to T$ and $\beta : T \to \widetilde{T}$ (we will use our isogeny β in the role of the isogeny that Shyr denotes by λ). Let β_v be the map of $\mathbb{Q}_v$-points induced by it at the place v, and β_v^c its restriction to T_v^c. Then [Shy77, Theorem 3] implies that:

$$\frac{h_T}{h_{T^{\mathrm{der}}}h_{\mathbb{G}_m}} = \frac{\tau_T}{\tau_{T^{\mathrm{der}}}\tau_{\mathbb{G}_m}}\frac{q(\beta_\infty)}{q(\beta_{\mathbb{Q}}(\infty))q((\hat{\beta})_{\mathbb{Q}})}\prod_\ell q(\beta_\ell^c). \tag{5.4}$$

(The right-hand side does not depend on the choice of the isogeny β since the left-hand side clearly does not depend on it.) Here $(\hat{\beta})_{\mathbb{Q}}$ is the map $(\hat{\widetilde{T}})_{\mathbb{Q}} \to (\hat{T})_{\mathbb{Q}}$ induced by β on $\mathbb{Q}$-rational characters, and $\beta_{\mathbb{Q}}(\infty)$ is the map $\widetilde{T}_{\mathbb{Q}}(\infty) \to T_{\mathbb{Q}}(\infty)$ induced by β on units. According to the last identity on

p.369 of *loc.cit.*, for all the tori we are interested in (where for T we have $r = r_\infty = 1$ and for T^{der}, $r = r_\infty = 0$), in fact $T_{\mathbb{Q}}(\infty) = T(\mathbb{Q}) \cap \mathbf{T}^c_{\mathbb{A}}$ – a finite group of order w_T.

We denote by Q the product $\prod_\ell q(\beta^c_\ell)$, which will cancel out later, and proceed.

Lemma 5.6. (a) $q((\hat{\beta})_{\mathbb{Q}}) = 1$.
(b) *We have* $T_{\mathbb{Q}}(\infty) = T^{\mathrm{der}}_{\mathbb{Q}}(\infty)$, *and thus* $w_T = w_{T^{\mathrm{der}}}$ *and* $q(\beta_{\mathbb{Q}}(\infty)) = 2$.
(c) $q(\beta_\infty) = 2$.

Proof. **(a)** We start by observing that the only characters of $\tilde{T}$ that are defined over $\mathbb{Q}$ are characters of $\mathbb{G}_m$, and the characters of T that are defined over $\mathbb{Q}$ are all the multiples of the multiplier character. By definition, for $t \in T$ and χ a $\mathbb{Q}$-character of $T^{\mathrm{der}} \times \mathbb{G}_m$ (i.e., a character of $\mathbb{G}_m$), $\hat{\beta}(\chi)(t) = \chi(\eta t)$, and thus $\hat{\beta}$ is surjective on rational characters; being a map of rank-1 lattices, it is injective.

(b) As shown in §2.2, $T(\mathbb{Q}) = \{a \in K^\times : a\bar{a} \in \mathbb{Q}^\times\}$, $T^{\mathrm{der}}(\mathbb{Q}) = \{a \in K^\times : a\bar{a} = 1\}$, where $\overline{(\cdot)}$ stands for the non-trivial Galois automorphism of K/K^+. At the same time, if an element of $a \in T(\mathbb{Q})$ is also in $\mathbf{T}^c$, this would imply in particular that $a\bar{a}$ is in $\mathbb{Z}_\ell^\times$ for all ℓ. (Indeed, $\eta \in (\hat{T})_{\mathbb{Q}} \subseteq (\hat{T})_{\mathbb{Q}_\ell}$.) Then $a\bar{a} = 1$ since it is both rational and an ℓ-adic unit for all ℓ. This immediately implies the equality $w_T = w_{T^{\mathrm{der}}}$ by the observations above this lemma. Finally, $\beta_{\mathbb{Q}}(\infty) : T_{\mathbb{Q}}(\infty) \to (T^{\mathrm{der}} \times \mathbb{G}_m)(\infty)$ is the map $t \mapsto (t/\eta(t), \eta t)$; where $t \in T_{\mathbb{Q}}(\infty) = T^{\mathrm{der}}_{\mathbb{Q}}(\infty)$. This map is surjective onto the set of elements $(t', 1) \in (T^{\mathrm{der}}_{\mathbb{Q}} \times \mathbb{G}_m)(\infty)$, but the pairs of the form $(t', -1)$ are not in its image, so its cokernel has order 2. At the same time this map is injective since when $\eta(t) = 1$ (which is the case for units), it is the identity on the first component.

(c). The map $\beta_\infty : T(\mathbb{R}) \to (T^{\mathrm{der}} \times \mathbb{G}_m)(\mathbb{R})$ is again defined by $\beta_\infty : a \mapsto (\bar{a}^{-1}, a\bar{a})$. This map again has cokernel of size 2 since $a\bar{a} > 0$, and is injective. We obtain $q(\beta_\infty) = 2$. □

We also record for future reference that (see equality (4) in [Shy77])

$$\mathrm{vol}_{\omega_{\tilde{T}}}(\tilde{T}^c_\ell) = \mathrm{vol}_{\omega_T}(T^c_\ell) q(\beta^c_\ell). \tag{5.5}$$

5.4. **The measure μ^{Shyr} vs. geometric measure.** Recall our notation: the measure ν_T gives the maximal compact subgroup T^c_ℓ of $T(\mathbb{Q}_\ell)$ volume 1, and we have denoted by μ^{Shyr} the quotient measure on the adjoint orbit of γ_ℓ, identified with $G(\mathbb{Q}_\ell)/T(\mathbb{Q}_\ell)$, which is obtained as the quotient of the canonical measure μ^{can}_G on $G(\mathbb{Q}_\ell)$ by the measure ν_T. The following proposition appears in [FLN10], where a similarly constructed measure is denoted by $\overline{\mu}_{T\backslash G,\ell}$. We observe that there are subtle differences in the definition of such a quotient measure related to the choice of a compact subgroup of $T(\mathbb{Q}_\ell)$ whose volume is set to be 1, but these differences do not matter in our version of the proposition, since the volume of the same compact subgroup appears in the denominator.

Proposition 5.7. *We have*

$$\mu^{\mathrm{geom}}_{\gamma,\ell} = |\eta(\gamma)|_\ell^{-\frac{g(g+1)}{4}} \sqrt{|D(\gamma)|_\ell} \frac{\mathrm{vol}_{\omega_G}(G(\mathbb{Z}_\ell))}{\mathrm{vol}_{\omega_T}(T^c_\ell)} \mu^{\mathrm{Shyr}}_\ell,$$

where $\eta(\gamma)$ is the multiplier of γ.

Proof. For $\gamma \in G^{\mathrm{der}}$, this follows from equation (3.30) of [FLN10]; we also reproved this relation in §4.2.1 of [AG17]. In fact, we derive there, for a general group G, the relation

$$\mu^{\mathrm{geom}}_{\gamma,\ell} = \sqrt{|D(\gamma)|_\ell} \frac{\mathrm{vol}_{\omega_G}(G(\mathbb{Z}_\ell))}{c_T} \mu^{can}_\ell,$$

where c_T is a constant that depends only on T, the centralizer of γ. As explained in [AG17, Lemma A.1], the constant c_T here is naturally the volume of the compact subgroup of the torus that is normalized to have volume 1; thus if we are dealing with the measure μ^{Shyr}, this should be the maximal compact subgroup T_ℓ^c. For general γ, the factor $|\eta(\gamma)|^{-\frac{g(g+1)}{4}}$ appears on the right-hand side, by considering the action of the centre of G on all the measure spaces involved. This is explained in detail in [Gor19]. □

We observe that for $\ell \neq p$, we have $|\eta(\gamma)|_\ell = |\det(\gamma)|_\ell = 1$; at p, we have $|\eta(\gamma)|_p = q^{-1}$, and we get the relation

$$\begin{aligned}
\nu_p([X,\lambda]) &= q^{-\frac{g(g+1)}{2}} \frac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{F}_p)} O^{\mathrm{geom}}_{\gamma_0}(\phi_{q,p}) \\
&= q^{-\frac{g(g+1)}{2}} q^{\frac{g(g+1)}{4}} \sqrt{|D(\gamma)|_p} \frac{\mathrm{vol}_{\omega_G}(G(\mathbb{Z}_\ell))}{\mathrm{vol}_{\omega_T}(T_\ell^c)} \frac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{F}_p)} O^{\mathrm{Shyr}}_{\gamma_0}(\phi_{q,p}) \qquad (5.6)\\
&= q^{-\frac{g(g+1)}{4}} \frac{\sqrt{|D(\gamma)|_p}}{\mathrm{vol}_{\omega_T}(T_\ell^c)} O^{\mathrm{Shyr}}_{\gamma_0}(\phi_{q,p}).
\end{aligned}$$

5.5. **Proof of Theorem 5.1.** Combining equality (5.3) with the results of §5.3, and using that $h_{\mathbb{G}_m} = \tau_{\mathbb{G}_m} = 1$, we get (changing the notation to $T' := T^{\mathrm{der}}$ to reduce the notational clutter):

$$\mathrm{vol}_{\nu_T}(T(\mathbb{Q})\backslash T(\mathbb{A}_f)) = \frac{h_T}{w_T} = \frac{h_{T'}}{w_{T'}} \frac{\tau_T}{\tau_{T'}\tau_{\mathbb{G}_m}} \frac{w_{T'}}{w_T} \frac{q(\beta_\infty)}{q(\beta_{\mathbb{Q}}(\infty))q((\hat{\beta})_{\mathbb{Q}})} Q = \frac{h_{T'}}{w_{T'}} \frac{\tau_T}{\tau_{T'}} Q.$$

Then by Proposition 5.7, we get (with the last equality coming from (5.5)):

$$\begin{aligned}
\mathrm{vol}_{\nu_T}(T(\mathbb{Q})\backslash T(\mathbb{A}_f)) \prod_\ell \mu_\ell^{\mathrm{Shyr}} &= \frac{h_{T'}}{w_{T'}} \frac{\tau_T}{\tau_{T'}} Q \prod_\ell \mu_\ell^{\mathrm{Shyr}} \\
&= \frac{h_{T'}}{w_{T'}} \frac{\tau_T}{\tau_{T'}} Q \prod_\ell \frac{\mathrm{vol}_{\omega_T}(T_\ell^c)}{\mathrm{vol}_{\omega_G}(G(\mathbb{Z}_\ell))\sqrt{|D(\gamma)|_\ell}} |\eta(\gamma)|_\ell^{\frac{g(g+1)}{4}} \mu^{\mathrm{geom}} \\
&= \frac{h_{T'}}{w_{T'}} \frac{\tau_T}{\tau_{T'}} Q |\eta(\gamma)|_\infty^{-\frac{g(g+1)}{4}} |D(\gamma)|_\infty^{1/2} Q^{-1} \prod_\ell \frac{\mathrm{vol}_{\omega_{\tilde{T}}}(\tilde{T}_\ell^c)}{\mathrm{vol}_{\omega_G}(G(\mathbb{Z}_\ell))} \mu^{\mathrm{geom}}.
\end{aligned}$$

Now, observe that $\mathrm{vol}_{\omega_{\tilde{T}}}(\tilde{T}_\ell^c) = \mathrm{vol}_{\omega_{T'}}(T'^c_\ell)\frac{\ell-1}{\ell}$, and therefore

$$\frac{\mathrm{vol}_{\omega_{\tilde{T}}}(\tilde{T}_\ell^c)}{\mathrm{vol}_{\omega_G}(G(\mathbb{Z}_\ell))} = \frac{\mathrm{vol}_{\omega_{T'}}(T'^c_\ell)}{\mathrm{vol}_{\omega_G^{\mathrm{der}}}(G^{\mathrm{der}}(\mathbb{Z}_\ell))}.$$

Now, using Shyr's formula (Theorem 5.3) for $h_{T'}/w_{T'}$, and then plugging in the expression for $D_{T'}$ in terms of L-factors from Lemma 5.5, we obtain:

$$\begin{aligned}
\mathrm{vol}_{\nu_T}(T(\mathbb{Q})\backslash T(\mathbb{A}_f))\prod_\ell \mu^{\mathrm{Shyr}} &= \frac{h_{T'}}{w_{T'}}\frac{\tau_T}{\tau_{T'}}|\eta(\gamma)|_\infty^{-\frac{g(g+1)}{4}}|D(\gamma)|_\infty^{1/2}\prod_\ell \frac{\mathrm{vol}_{\omega_{T'}}(T'^c_\ell)}{\mathrm{vol}_{\omega_G^{\mathrm{der}}}(G^{\mathrm{der}}(\mathbb{Z}_\ell))}\mu^{\mathrm{geom}} \\
&= \frac{\tau_T}{\tau_{T'}}\frac{|\eta(\gamma)|_\infty^{-\frac{g(g+1)}{4}}|D(\gamma)|_\infty^{1/2}\rho_{T'}\tau_{T'}|D_{T'}|^{1/2}}{R_{T'}}\prod_\ell \frac{\mathrm{vol}_{\omega_{T'}}(T'^c_\ell)}{\mathrm{vol}_{\omega_G^{\mathrm{der}}}(G^{\mathrm{der}}(\mathbb{Z}_\ell))}\mu^{\mathrm{geom}} \\
&= \frac{\tau_T}{\tau_{T'}}\frac{|\eta(\gamma)|_\infty^{-\frac{g(g+1)}{4}}|D(\gamma)|_\infty^{1/2}\tau_{T'}L(1,\sigma_{T'})}{(2\pi)^g\prod_\ell L_\ell(1,\sigma_{T'})\,\mathrm{vol}_{\omega_{T'}}(T'^c_\ell)}\prod_\ell \frac{\mathrm{vol}_{\omega_{T'}}(T'^c_\ell)}{\mathrm{vol}_{\omega_{\mathrm{Sp}_{2g}}}(\mathrm{Sp}_{2g}(\mathbb{Z}_\ell))}\mu^{\mathrm{geom}} \\
&= |\eta(\gamma)|_\infty^{-\frac{g(g+1)}{4}}\frac{|D(\gamma)|_\infty^{1/2}\tau_T}{(2\pi)^g}\prod \frac{1}{\mathrm{vol}_{\omega_{\mathrm{Sp}_{2g}}}(\mathrm{Sp}_{2g}(\mathbb{Z}_\ell))}\mu^{\mathrm{geom}}.
\end{aligned}$$

The theorem now follows from the simple observation that

$$\mathrm{vol}_{\omega_{\mathrm{Sp}_{2g}}}(\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)) = \frac{\#\mathrm{Sp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})}{\ell^{\dim(\mathrm{Sp}_{2g})}}.$$

5.6. **Proof of Theorem A.** Now we are ready to prove the main theorem. Recall that given $[X,\lambda]/\mathbb{F}_q$, we have chosen an element $\gamma_0 \in G(\mathbb{A}_f)$, and a $\delta_0 \in G(\mathbb{Q}_q)$, and used them to define local ratios ν_ℓ. The claim to be proved is that

$$\tilde{\#}I([X,\lambda],\mathbb{F}_q) = q^{\frac{g(g+1)}{4}}\tau_T\nu_\infty([X,\lambda])\prod_\ell \nu_\ell([X,\lambda]).$$

We prove this by showing that the product of local probabilities on the right-hand side equals the right-hand side of the formula of Langlands and Kottwitz.

Combining Corollaries 4.7, 4.12, and Theorem 5.1, and plugging in $|\eta(\gamma_0)|_\infty = q$ (see also (5.6) for the combined form of the most complicated term at $\ell = p$), we get:

$$\begin{aligned}
\prod_\ell \nu_\ell(\gamma_0) = \nu_p(\gamma_0)\prod_{\ell\neq p}\nu_\ell(\gamma_0) &= q^{-\frac{g(g+1)}{2}}O^{\mathrm{geom}}_{\gamma_0}(\phi_{q,p})\left(\prod_{\ell\neq p}O^{\mathrm{geom}}_{\gamma_0}(\phi_{0,\ell})\right)\left(\prod_\ell \frac{\ell^{\dim(\mathrm{Sp}_{2g})}}{\#\mathrm{Sp}(\mathbb{Z}/\ell)}\right) \\
&= q^{-\frac{g(g+1)}{4}}\frac{(2\pi)^g}{|D(\gamma)|_\infty^{1/2}\tau_T}\,\mathrm{vol}^{\mathrm{Shyr}}(T(\mathbb{Q})\backslash T(\mathbb{A}^f))\prod_{\ell\neq p}O^{\mathrm{Shyr}}(\phi_{0,\ell})O^{\mathrm{Shyr}}(\phi_{q,p}),
\end{aligned}$$

which is exactly the right-hand side of the Langlands and Kottwitz formula up to the factor $q^{\frac{-g(g+1)}{4}}\frac{(2\pi)^g}{|D(\gamma)|_\infty^{1/2}\tau_T}$ that we need to take to the other side.

6. Complements

For the convenience of a hypothetical reader interested in explicit calculations, we collect here some reminders concerning the terms which arise in (1.3).

6.1. ν_∞**.** Recall that we have defined (4.2) $\nu_\infty([X,\lambda])$ as $\sqrt{|D(\gamma_0)|}/(2\pi)^g$, where $D(\gamma_0)$ is the Weyl discriminant $D(\gamma_0) = \prod_{\alpha\in\Phi}(1-\alpha(\gamma_0))$, the product being over all roots of G. We may relate this to the (polynomial) discriminant of $f_{X/\mathbb{F}_q}(T)$, the characteristic polynomial of Frobenius, as follows.

6.1.1. *Weyl discriminants.* Explicitly, γ_0 has multiplier $\lambda_0 := \eta(\gamma_0) = q$. Write the (complex) eigenvalues of (a $\mathbb{Q}$-representative of) γ_0 – equivalently, the roots of $f_{X/\mathbb{F}_q}(T)$ – as $(\lambda_1, \cdots, \lambda_g, \lambda_0/\lambda_1, \cdots, \lambda_0/\lambda_g)$. Then

$$D(\gamma) = \prod_{1\le i<j\le g} \delta_{ij} \cdot \prod_{1\le i\le g} \delta_i$$

where

$$\begin{aligned}\delta_{ij} &= (1-\lambda_i/\lambda_j)(1-\lambda_j/\lambda_i)(1-\lambda_i\lambda_j/\lambda_0)(1-\lambda_0/(\lambda_i\lambda_j))\\ \delta_i &= (1-\lambda_i^2/\lambda_0)(1-\lambda_0/\lambda_i^2).\end{aligned}$$

Possibly after reordering the conjugate pairs $\{\lambda_i, \lambda_0/\lambda_i\}$, we may and do assume that $\lambda_j = \sqrt{q}\exp(i\theta_j)$ with $0 \le \theta_j < \pi$. Then

$$\begin{aligned}\delta_{ij} &= (2\cos(\theta_i) - 2\cos(\theta_j))^2\\ \delta_j &= 4\sin^2(\theta_j).\end{aligned}$$

6.1.2. *Elliptic curves.* Suppose that $[X, \lambda]$ is an elliptic curve with its canonical principal polarization, say with characteristic polynomial of Frobenius $T^2 - aT + q$. Then $a = 2\sqrt{q}\cos(\theta)$, and $D(\gamma_0) = 4\sin^2(\theta) = 4 - \frac{a^2}{q}$, and

$$\nu_\infty([X,\lambda]/\mathbb{F}_q) = \frac{1}{2\pi}\sqrt{|D(\gamma_0)|} = \frac{1}{\pi}\sqrt{1-\frac{a^2}{4q}}.$$

Note that this term is *half* the archimedean term introduced in [Gek03, (3.3)] (when $q = p$) and [AG17, (2-7)]. For purposes of comparison, we summarize this relationship by writing

$$\nu_\infty([X,\lambda]) = \frac{1}{2}\nu_\infty^{\mathrm{Gek}}([X,\lambda]) = \frac{1}{2}\nu_\infty^{\mathrm{AG}}([X,\lambda]).$$

6.1.3. *Polynomial discriminants.* To facilitate comparison with [AW15, Gek03, GW19], we express $D(\gamma_0)$ in terms of polynomial discriminants. Let $f(T) = f_{X/\mathbb{F}_q}(T)$, and let $f^+(T) = f^+_{X/\mathbb{F}_q}(T)$ be the minimal polynomial of the sum of γ_0 and its adjoint, so that

$$f^+_{X/\mathbb{F}_q}(T) = \prod_{1\le j\le g}(T - (\lambda_j + q/\lambda_j)).$$

Note that $\mathbb{Q}[T]/f^+(T) \cong K^+$, the maximal totally real subalgebra of the endomorphism algebra of X.

Lemma 6.1. *We have*

$$\frac{\mathrm{disc}(f(T))}{\mathrm{disc}(f^+(T))} = (-1)^g q^{\frac{g(3g-1)}{2}} D(\gamma_0).$$

Proof. On one hand,

$$\mathrm{disc}(f(T)) = \prod_{1\le i<j\le g}\alpha_{ij}^2 \prod_{1\le i\le g}\alpha_i^2$$

where

$$\alpha_{ij} = (\lambda_i - \lambda_j)(\lambda_i - \lambda_0/\lambda_j)(\lambda_0/\lambda_i - \lambda_j)(\lambda_0/\lambda_i - \lambda_0/\lambda_j)$$

and

$$\alpha_i = (\lambda_i - \lambda_0/\lambda_i).$$

On the other hand,

$$\operatorname{disc}(f^+(T)) = \prod_{1\le i<j\le g} \beta_{ij}^2$$

where

$$\beta_{ij} = (\lambda_i + \lambda_0/\lambda_i - (\lambda_j + \lambda_0/\lambda_j)).$$

Now use this to evaluate $\operatorname{disc}(f(T))/\operatorname{disc}(f^+(T))$, while bearing in mind that

$$\frac{\alpha_{ij}}{\beta_{ij}^2} = \lambda_0 \text{ and } \frac{\alpha_{ij}}{\delta_{ij}} = \lambda_0^2 \text{ and } \frac{\alpha_i^2}{\delta_i} = -\lambda_0.$$

□

6.2. ν_ℓ. Gekeler [Gek03] observed that, for elliptic curves, his product formula essentially computes an L-function; a similar phenomenon has been observed in other contexts, as well [AW15, GW19]. We briefly explain how this relates to (1.3). This detour also has the modest benefit of showing that the right-hand side of (1.3) converges, albeit conditionally.

6.2.1. *Zeta functions.* We express the zeta function of a number field M as $\zeta_M(s) = \prod_\ell \zeta_{M,\ell}(s)$, where $\zeta_{M,\ell}(s) = \prod_{\lambda|\ell}(1 - \mathrm{N}_{M/\mathbf{Q}}(\lambda)^{-s})^{-1}$. For a direct sum $M = \oplus_{i=1}^t M_i$ of such fields we write $\zeta_{M,\ell}(s) = \prod_i \zeta_{M_i,\ell}(s)$; the product over all primes yields $\zeta_M(s) = \prod_i \zeta_{M_i}(s)$.

Recall (as in §5.1.2) that to a torus $S/\mathbf{Q}$ one associates an Artin L-function $L(s,\sigma_S) = \prod_\ell L_\ell(s,\sigma_S)$. This construction is multiplicative for exact sequences of tori, and for a finite direct sum M of number fields one has $L(s,\sigma_{\mathbf{R}_{M/\mathbf{Q}}\mathbb{G}_m}) = \zeta_M(s)$. (It may be worth recalling that $\mathbf{R}_{M/\mathbf{Q}}\mathbb{G}_m \cong \oplus\, \mathbf{R}_{M_i/\mathbf{Q}}\mathbb{G}_m$.)

If ℓ is unramified in some splitting field for S, then (cf. [Bit11, 2.8], [Vos98, 14.3]) one has

$$\#S(\mathbb{F}_\ell) = \ell^{\dim S} L_\ell(1,\sigma_S)^{-1}.$$

Lemma 6.2. *Suppose that $\ell \nmid 2p\operatorname{disc}(f_{X/\mathbb{F}_q}(T))$. Then*

$$\nu_\ell([X,\lambda]) = \frac{\zeta_{K,\ell}(1)}{\zeta_{K^+,\ell}(1)}.$$

Proof. By Lemma 4.4

$$\begin{aligned}\nu_\ell([X,\lambda]) &= \frac{\#\{\gamma \in G(\mathbb{F}_\ell) : \gamma \sim \pi_1(\gamma_0)\}}{\#G(\mathbb{F}_\ell)/\#\mathbb{A}_G(\mathbb{F}_\ell)} \\ &= \frac{\#G(\mathbb{F}_\ell)/\#T(\mathbb{F}_\ell)}{\#G(\mathbb{F}_\ell)/\left(\ell^g \#\mathbb{G}_m(\mathbb{F}_\ell)\right)} \\ &= \ell^g \frac{\#\mathbb{G}_m(\mathbb{F}_\ell)}{\#T(\mathbb{F}_\ell)} = \frac{L_\ell(1,\sigma_T)}{L_\ell(1,\sigma_{\mathbb{G}_m})},\end{aligned}$$

since $\dim T = g+1$. Using (2.2), first to see that $L(s,\sigma_T) = L(s,\sigma_{T^{\mathrm{der}}})L(s,\sigma_{\mathbb{G}_m})$ and second to compute $L(s,\sigma_{T^{\mathrm{der}}})$, we recognize this as

$$= \frac{\zeta_{K,\ell}(1)}{\zeta_{K^+,\ell}(1)}.$$

□

Since $\zeta_K(s)$ and $\zeta_{K^+}(s)$ both have a simple pole at $s=1$, we immediately deduce:

Corollary 6.3. *The right-hand side of* (1.3) *converges conditionally.*

Moreover, up to a finite factor $B([X,\lambda])$, we can express $\widetilde{\#}I([X,\lambda],\mathbb{F}_q)$ in terms of familiar quantities:

Corollary 6.4. *We have*

$$\widetilde{\#}I([X,\lambda],\mathbb{F}_q) = \tau_T \frac{q^{-\frac{g(3g-1)}{4}}}{(2\pi)^g}\sqrt{\left|\frac{\operatorname{disc}(f)}{\operatorname{disc}(f^+)}\right|}B([X,\lambda]) \lim_{s\to 1^+}\frac{\zeta_K(s)}{\zeta_{K^+}(s)}$$

where

$$B([X,\lambda]) = \prod_{\ell \mid 2p\operatorname{disc}(f)} \frac{\zeta_{K^+,\ell}(1)}{\zeta_{K,\ell}(1)}\nu_\ell([X,\lambda]).$$

6.2.2. *Elliptic curves.* If $[X,\lambda]$ is an elliptic curve, let χ be the quadratic character associated to the imaginary quadratic field K. Then $K^+ = \mathbb{Q}$, and $\zeta_K(s)/\zeta_{\mathbb{Q}}(s) = L(s,\chi)$.

Now further suppose that $q = p$ and that the Frobenius order is maximal, i.e., that $\mathbb{Z}[T]/f_{X/\mathbb{F}_q}(T) \cong \mathcal{O}_K$. Then Gekeler shows with an explicit calculation that for *each* prime ℓ, $\nu_\ell^{\mathrm{Gek}}([X,\lambda]) = L_\ell(1,\chi)$, and thus $\prod_\ell \nu_\ell^{\mathrm{Gek}}([X,\lambda]) = L(1,\chi)$.

6.2.3. *Abelian varieties with maximal Frobenius order.* Similarly, suppose $[X,\lambda]$ is an ordinary abelian surface with $\operatorname{End}(X)\otimes\mathbb{Q}$ a cyclic quartic extension of $\mathbb{Q}$, and further suppose that the Frobenius order is maximal. In [AW15], the authors define a local term $\nu_\ell^{\mathrm{AW}}([X,\lambda])$, and show that $\prod_\ell \nu_\ell^{\mathrm{AW}}([X,\lambda]) = \zeta_K(1)/\zeta_{K^+}(1)$. This observation has been extended to certain abelian varieties of prime dimension [GW19, Prop. 8.1].

6.3. ν_p**.** Since the multiplier $\eta(\gamma_0)$ of Frobenius is q, γ_0, while an element of $M(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$, is never an element of $G(\mathbb{Z}_p)$. Nonetheless, if the isogeny class is *ordinary*, it is possible to transfer part of the work in calculating $\nu_p([X,\lambda])$ from $M(\mathbb{Z}_p)$ to $G(\mathbb{Z}_p)$, as follows.

Suppose X is ordinary. Then its p-divisible group splits *integrally* as $X[p^\infty] = X[p^\infty]^{\mathrm{tor}} \oplus X[p^\infty]^{\text{ét}}$. (In general, the slope filtration only exists up to isogeny, as in Lemma 3.5.) Therefore, there exists a decomposition $V_{\mathbb{Z}_p} = V_{\mathbb{Z}_p}^{\text{ét}} \oplus V_{\mathbb{Z}_p}^{\mathrm{tor}}$ into maximal isotropic summands stable under γ_0, where $\alpha_0 := \gamma_0|_{V_{\mathbb{Z}_p}^{\text{ét}}} \in \operatorname{End}(V_{\mathbb{Z}_p}^{\text{ét}})$ is invertible; and the polarization induces an isomorphism $V_{\mathbb{Z}_p}^{\mathrm{tor}}$ with the dual of $V_{\mathbb{Z}_p}^{\text{ét}}$, such that $\gamma_0|_{V_{\mathbb{Z}_p}^{\mathrm{tor}}} = q(\alpha_0^\top)^{-1}$. This can also be proved directly through linear algebra. Indeed, let $\beta_0 = q\gamma_0^{-1}$. Then $V_{\mathbb{Z}_p}^{\text{ét}} = \cap_n \gamma_0^{\circ n}(V_{\mathbb{Z}_p})$, while $V_{\mathbb{Z}_p}^{\mathrm{tor}} = \cap_n \beta_0^{\circ n}(V_{\mathbb{Z}_p})$.

Lemma 6.5. *For n and d sufficiently large and $\gamma \in M(\mathbb{Z}_p/p^n)$, the following conditions are equivalent:*

(a) $\gamma \sim_{M(\mathbb{Z}_p/p^n)_d} \gamma_0 \bmod p^n$;
(b) *there exists some $\widetilde{\gamma} \in M(\mathbb{Z}_p)$ such that $\widetilde{\gamma} \bmod p^n = \gamma$ and $\widetilde{\gamma} \sim_{G(\mathbb{Q}_p)} \gamma_0$;*
(c) *γ stabilizes a decomposition $V_{\mathbb{Z}_p/p^n} \cong V^+_{\mathbb{Z}_p/p^n} \oplus V^-_{\mathbb{Z}_p/p^n}$ into maximal isotropic subspaces, and there exists an isomorphism $\iota : V^+_{\mathbb{Z}_p/p^n} \to V^{\text{ét}}_{\mathbb{Z}_p/p^n}$ such that $\iota^*\alpha_0 = \gamma|_{V^+_{\mathbb{Z}_p/p^n}}$.*

Proof. The equivalence of (a) and (b) is Lemma 3.2. For the equivalence of (a) and (c), use the argument above to show that any such γ induces an appropriate decomposition of $V_{\mathbb{Z}_p/p^n}$. □

Therefore, if $\alpha_0 \bmod p$ is regular, we obtain a version of Lemma 6.2 at p.

Corollary 6.6. *Suppose* $[X, \lambda]$ *is ordinary and* $\operatorname{ord}_p \operatorname{disc}(f_{X/\mathbb{F}_q}(T)) = e \cdot g(g-1)$. *Then*

$$\nu_p([X, \lambda]) = \frac{\zeta_{K,p}(1)}{\zeta_{K^+,p}(1)}.$$

Note that we always have $q^{g(g-1)} | \operatorname{disc}(f_{X/\mathbb{F}_q}(T))$. The case $g = 1$ also follows from the explicit calculation in [Gek03, Thm. 4.4].

Proof. Define $\epsilon_0 \in \operatorname{Sp}(V_{\mathbb{Z}_p})$ by $\epsilon_0|_{V^{\text{ét}}_{\mathbb{Z}_p}} = \alpha_0$ and $\epsilon_0|_{V^{\text{tor}}_{\mathbb{Z}_p}} = (\alpha_0^\top)^{-1}$.

The argument of Lemma 6.5 shows that, for sufficiently large d and n, both $\#C_{(d,n)}(\gamma_0)$ and $\#C_{(d,n)}(\gamma_0)$ are given by the product of the number of decompositions $V_{\mathbb{Z}_p/p^n} = V^+_{\mathbb{Z}_p/p^n} \oplus V^-_{\mathbb{Z}_p/p^n}$ into maximal isotropic subspaces, and the number of $\alpha \in \operatorname{End}(V_{\mathbb{Z}_p/p^n})$ with $\alpha \sim_{\operatorname{End}(V_{\mathbb{Z}_p/p^n})_d} \alpha_0$. In particular, $\#C_{(d,n)}(\gamma_0) = \#C_{(d,n)}(\epsilon_0)$.

The regularity hypothesis implies that $\epsilon_0 \bmod p$ is regular, and the result follows from Lemma 6.2. □

6.4. **Explicit examples.**

6.4.1. $g = 1$. Consider the elliptic curve $E/\mathbb{F}_7$ with affine equation $y^2 = x^3 + x + 1$. Its Frobenius polynomial is $f_E(T) = T^2 - 3T + 7$, which has discriminant -19, a fundamental discriminant. So the order generated by the Frobenius endomorphism is the ring of integers in $K := \mathbb{Q}(\sqrt{-19})$; using Magma, we numerically estimate $\prod_\ell \nu_\ell(E/\mathbb{F}_7) = L(1, \left(\frac{-19}{\bullet}\right))$ as ≈ 0.72073. Continuing to work numerically, we have $\nu_\infty(E/\mathbb{F}_7) = \frac{1}{2\pi}\sqrt{4 - \frac{3^2}{7}} \approx 0.2622$. Since $T = \mathbf{R}_{K/\mathbb{Q}}\mathbb{G}_m$ we have $\tau_T = 1$, and thus $\tau_T\sqrt{7}\nu_\infty(E/\mathbb{F}_7)\prod_\ell \nu_\ell(E/\mathbb{F}_7) \approx 0.5000$.

This reflects the easily verified arithmetic statement that the only elliptic curve over $\mathbb{F}_7$ with trace of Frobenius 3 is E itself; and $\operatorname{Aut}(E) \cong \mathcal{O}_K^\times = \{\pm 1\}$, so that the weighted size of this isogeny class is $\widetilde{\#} I([E], \mathbb{F}_7) = \frac{1}{2}$. (In modest contrast, [Gek03] assigns weight $2/\#\operatorname{Aut}(F)$ to an elliptic curve F; this is reflected in the fact that $\nu_\infty^{\text{Gek}}([E], /\mathbb{F}_7) = 2\nu_\infty([E], \mathbb{F}_7)$.)

6.4.2. $g = 4$. Consider the 3-Weil polynomial

$$f(T) = T^8 - 6T^7 + 13T^6 - 10T^5 + T^4 - 30T^3 + 117T^2 - 162T + 81.$$

It turns out that there is a unique principally polarized abelian fourfold (X, λ) over $\mathbb{F}_3$ with characteristic polynomial equal to $f(T)$. (This is a single datapoint in a census of isogeny classes which will soon be integrated into the LMFDB.)

Let $K = \mathbb{Q}[T]/f(T)$. One readily checks that $\operatorname{disc}(f(T))/\operatorname{disc}(K) = 3^{4(4-1)}$, and so $\nu_\ell([X, \lambda]) = \zeta_{K,\ell}(1)/\zeta_{K^+,\ell}(1)$ for all finite ℓ, including $\ell = p$. Again, we numerically estimate $\prod_\ell \nu_\ell([X, \lambda]) = \lim_{s \to 1^+} \frac{\zeta_K(s)}{\zeta_{K^+}(s)} \approx 0.871253$ and $\nu_\infty([X, \lambda]) \approx 0.000111808$. The field K is Galois over $\mathbb{Q}$, with group $\operatorname{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$, and the Tamagawa number of the torus T is 2 (see A.6). Our formula numerically yields

$$\widetilde{\#} I([X, \lambda], \mathbb{F}_3) = 3^3 \tau_T \nu_\infty([X, \lambda], \mathbb{F}_3) \lim_{s \to 1} \frac{\zeta_K(s)}{\zeta_{K^+}(s)} \approx 0.050000.$$

This reflects the fact that the torsion group of $\mathcal{O}_K^\times$, and thus $\operatorname{Aut}([X, \lambda])$, has order 20.

6.5. **Level structure.** The Langlands-Kottwitz formula (2.1) is actually written for abelian varieties with arbitrary level structure, and thus a version of our main formula is available in the context of abelian varieties with level structure, too.

6.5.1. *Product formula.* Let $\Gamma \subset G(\hat{\mathbb{Z}}_f^p)$ be an open compact subgroup. There is a notion of principally polarized abelian variety with level Γ structure; let $\mathcal{A}_{g,\Gamma}$ be the corresponding Shimura variety. If $(X, \lambda, \alpha) \in \mathcal{A}_{g,\Gamma}(\mathbb{F}_q)$ is a principally polarized abelian variety with level Γ-structure, then the size of its isogeny class in this category is given by the Kottwitz formula, except that the integrand in the adelic orbital integral is replaced with $\mathbb{1}_\Gamma$.

We make the definition

$$\nu_\ell([X, \lambda, \alpha]) = \lim_{d\to\infty} \lim_{n\to\infty} \frac{\#(C_{(d,n)}(\gamma_0) \cap \pi_n(\Gamma_\ell))}{\#G(\mathbb{Z}_\ell/\ell^n)/\#\mathbb{A}_G(\mathbb{Z}_\ell/\ell^n)}.$$

The analogue of Corollary 4.6 holds, and states that there exists $d(\gamma_0)$ such that

$$O_{\gamma_0}^{\mathrm{geom}}(\mathbf{1}_{\Gamma_\ell}) = \lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_G|}(\widetilde{C}_{(d(\gamma_0),n)}(\gamma_0) \cap \Gamma_\ell)}{\mathrm{vol}_{|d\omega_A|}(\widetilde{U}_n(\gamma_0))}.$$

The calculations at p and ∞, as well as the global volume term, are unchanged, and we find that

$$\tilde{\#}I([X, \lambda, \alpha], \mathbb{F}_q) = q^{\frac{g(g+1)}{4}} \tau_T \nu_\infty([X, \lambda]) \prod_\ell \nu_\ell([X, \lambda, \alpha]). \tag{6.1}$$

6.5.2. *Principal level structure.* Fix a prime ℓ_0, and define $\Gamma(\ell_0) = \prod_\ell \Gamma(\ell_0)_\ell$ by

$$\Gamma(\ell_0)_\ell = \begin{cases} G(\mathbb{Z}_\ell) & \ell \neq \ell_0 \\ \ker(G(\mathbb{Z}_{\ell_0}) \to G(\mathbb{Z}_{\ell_0}/\ell_0)) & \ell = \ell_0. \end{cases}$$

Then $\mathcal{A}_{g,\Gamma(\ell_0)}$ is the moduli space of abelian varieties equipped with a full principal level ℓ_0-structure.

For example, to fix ideas, suppose that $g = 1$ and that $\ell_0 \neq p$, and let a satisfy $|a| \leq 2\sqrt{q}$, $p \nmid a$ and $\ell_0 \| (a^2 - 4q)$; we consider the set of elliptic curves with characteristic polynomial of Frobenius $f(T) = T^2 - aT + q$. Then some, but not all, elements of the corresponding isogeny class admit a principal level ℓ_0-structure (see, e.g., [AW13]).

Let (X, λ, α) be an elliptic curve over $\mathbb{F}_q$ with trace of Frobenius a and full level ℓ_0-structure α. We may explicitly compute $\nu_{\ell_0}([X_0, \lambda, \alpha])$ as follows. Let $\chi_{\ell_0} = \left(\frac{\cdot}{\ell_0}\right)$ be the quadratic character modulo ℓ_0.

Lemma 6.7. *We have*

$$\nu_{\ell_0}([X, \lambda, \alpha]) = \frac{1}{\ell_0^2} \frac{1}{1 - \chi_{\ell_0}(\mathrm{disc}(f)/\ell_0^2)/\ell_0}.$$

Proof. Let $\gamma_0 = \gamma_{X,\mathbb{F}_q,\ell_0}$ be a Frobenius element for X at ℓ_0. By hypothesis, $\gamma_0 = 1 + \ell_0\beta_0$ for some $\beta_0 \in \mathrm{Mat}_2(\mathbb{Z}_{\ell_0})$. Since ℓ_0^2 is the highest power of ℓ_0 dividing $\mathrm{disc}(f)$, we in fact have $\beta_0 \in \mathrm{GL}_2(\mathbb{Z}_{\ell_0})$, and β_0 is regular $\bmod \ell_0$, i.e., $\pi_1(\beta_0)$ is regular.

Suppose that $\gamma \in \Gamma_{\ell_0}$ satisfies $\gamma \sim_{G(\mathbb{Q}_{\ell_0})} \gamma_0$. Then $\gamma = 1 + \ell_0\beta$ for some $\beta \in \mathrm{GL}_2(\mathbb{Z}_{\ell_0})$ which is regular $\bmod \ell_0$, and direct calculation shows $\beta \sim_{G(\mathbb{Q}_{\ell_0})} \beta_0$. Lemma 3.1 then shows that $\beta \sim_{G(\mathbb{Z}_\ell)} \beta_0$.

Consequently, for any $d \geq 0$ and any $n \geq 2$, we have bijections between the following sets:

$$\{\gamma \in G(\mathbb{Z}_{\ell_0}/\ell_0^n) : \gamma \sim_{M(\mathbb{Z}_{\ell_0}/\ell_0^n)_d} \pi_n(\gamma_0)\};$$
$$\{\beta \in G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1}) : \beta \sim_{M(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})_d} \pi_{n-1}(\beta_0)\};$$
$$\text{and } \{\beta \in G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1}) : \beta \sim_{G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})} \pi_n(\beta_0)\}.$$

Therefore,

$$\begin{aligned}
\nu_{\ell_0}([X,\lambda,\alpha]) &= \frac{\#G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})/\#G_{\beta_0}(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})}{\#G(\mathbb{Z}_{\ell_0}/\ell_0^{n})/\#\mathbb{A}_G(\mathbb{Z}_{\ell_0}/\ell_0^{n})} \\
&= \frac{\#G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})}{\#G(\mathbb{Z}_{\ell_0}/\ell_0^{n})}\frac{\#\mathbb{A}_G(\mathbb{Z}_{\ell_0}/\ell_0^{n})}{\#G_{\beta_0}(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})} \\
&= \frac{1}{\ell_0^4}\frac{\ell_0^2\#\mathbb{A}_G(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})}{\#G_{\beta_0}(\mathbb{Z}_{\ell_0}/\ell_0^{n-1})} \\
&= \frac{1}{\ell_0^2}\frac{\#\mathbb{A}_G(\mathbb{Z}_{\ell_0}/\ell_0)}{\#G_{\beta_0}(\mathbb{Z}_{\ell_0}/\ell_0)} \\
&= \frac{1}{\ell_0^2}\frac{1}{1-\chi_{\ell_0}(\mathrm{disc}(f)/\ell_0^2)/\ell_0}.
\end{aligned}$$

□

7. GL_2 RECONSIDERED

In [AG17], we essentially treated the $g = 1$ case of the present paper. Unfortunately, a simple algebra error – $\nu_\infty^{\mathrm{AG}}([X,\lambda]) = \frac{2}{\pi}\sqrt{|D(\gamma_0)|}$ (§6.1.2), in spite of the claims of the penultimate displayed equation [AG17, p.20] – masked certain mistakes involving the calculations at p. We take the opportunity to correct these mistakes. The reader pleasantly unaware of these issues with [AG17] may simply view the present section as an explication of our technique in the special case where $g = 1$, and thus $G = \mathrm{GL}_2$.

Note that the definition [AG17, (2-6)] could have been replaced with a criterion involving characteristic polynomials, e.g.,

$$\nu_p(a,q) = \lim_{n\to\infty}\frac{\#\{\gamma \in \mathrm{Mat}_2(\mathbb{Z}_p/p^n) : f_\gamma \equiv f_{\gamma_0} \bmod p^n\}}{\#G(\mathbb{Z}_p/p^n)/\#A(\mathbb{Z}_p/p^n)}.$$

7.1. **Assertions at p.** There are two problematic claims in [AG17]:

(1) For the test function $\mathbb{1}_{G(\mathbb{Z}_\ell)}$ at $\ell \neq p$, we have

$$\nu_\ell(a,q) = \frac{\ell^3}{\#\,\mathrm{SL}_2(\mathbb{F}_\ell)}\mathcal{O}_{\gamma_0}^{\mathrm{geom}}(\mathbb{1}_{G(\mathbb{Z}_\ell)}).$$

It is claimed in [AG17, Lemma 3.7] that the same is true for $\ell = p$, where the test function ϕ_q is the characteristic function of $G(\mathbb{Z}_p)\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}G(\mathbb{Z}_p)$.

(2) In [AG17, Appendix], revisiting the calculation [FLN10, (3.30)], we assert that.

$$\mu^{\mathrm{geom}}_{\gamma,\ell} = \sqrt{|D(\gamma)|_\ell}\frac{\mathrm{vol}_{|\omega_G|_\ell}(G(\mathbb{Z}_\ell))}{\mathrm{vol}_{|\omega_T|_\ell}(T^\circ(\mathbb{Z}_\ell))}\overline{\mu}_{T\backslash G,\ell}. \tag{7.1}$$

This is valid for $\ell \neq p$, but requires correction at p, because our Chevalley-Steinberg map is not exactly the same as the map in [FLN10].

7.2. **From point-counts to measure.** In (1), one exploits the fundamental fact that point counts $\mathrm{mod}\,\ell^n$ converge to volume with respect to the Serre-Oesterlé measure μ^{SO}. In the case where $\ell = p$, however, the ambient space is $\mathrm{Mat}_2(\mathbb{Z}_p)$, rather than (its open subset) $G(\mathbb{Z}_p)$. Thus, the volume of the set V_n of [AG17] should be computed with respect to $\mu^{\mathrm{SO}}_{\mathrm{Mat}_2}$.

Moreover, one should be using the invariant measure $dx \wedge \frac{dy}{|y|}$ on the Steinberg base $\mathbb{A}_{\mathrm{GL}_2} \cong \mathbb{A}^1 \times \mathbb{G}_m$, rather than the measure pulled back from $\mathbb{A}^1 \times \mathbb{A}^1$. Then the measure of a radius p^{-n}-neighborhood U_n of (a,q) in A is $p^{-2n}/|\det(\gamma_0)| = p^{-2n}/q^{-1}$, and we find

$$\begin{aligned}\nu_{p,n}(a,q) &= \frac{p^3}{\#\mathrm{SL}_2(\mathbb{F}_p)}\frac{|\det(\gamma_0)|^2\,\mathrm{vol}_{\mu_{GL_2}}(V_n(\gamma_0))}{|\det(\gamma_0)|\,\mathrm{vol}_{\mathbb{A}^1\times\mathbb{G}_m}(U_n)} \\ &= |\det(\gamma_0)|\frac{p^3}{\#\mathrm{SL}_2(\mathbb{F}_p)}O^{\mathrm{geom}}_{\gamma_0}(\phi_q) = q^{-1}\frac{p^3}{\#\mathrm{SL}_2(\mathbb{F}_p)}O^{\mathrm{geom}}_{\gamma_0}(\phi_q).\end{aligned}$$

(This differs from the assertion of [AG17, Lemma 3.7] by a factor of q.)

7.3. **From geometric measure to canonical measure.** Since orbital integrals of rational-valued functions with respect to the canonical measure are rational, while $\sqrt{|D(\gamma_0)|_p} = \sqrt{q}$, the assertion of (2) cannot hold at $\ell = p$.

While the relation between the geometric measure and the canonical measure that we rely on is correct for a semisimple group, it needs a correction factor for a reductive group. This part is completely general for all reductive G, and is discussed in detail in [Gor19]; the correct formula is stated in Proposition 5.7. In particular, the correct calculation at p is

$$\mu^{\mathrm{geom}}_{\gamma,p} = |\eta(\gamma)|_p^{-\frac{g(g+1)}{4}}\sqrt{|D(\gamma)|_p}\frac{\mathrm{vol}_{|\omega_G|_p}(G(\mathbb{Z}_p))}{\mathrm{vol}_{|\omega_T|_p}(T^\circ(\mathbb{Z}_p))}\overline{\mu}_{T\backslash G,p},$$

where $\eta(\gamma)$ is the multiplier of γ.

Appendix A. By Wen-Wei Li and Thomas Rüd

We compute the Tamagawa numbers of some anisotropic tori in GSp_{2g} and Sp_{2g} associated with a single Galois field extension (see section 2.2), and present a partial result towards the general case that illustrates the difficulties.

Recall the setup of 2.2 in the case of a single Galois extension. Let $K \supset K^+ \supset \mathbb{Q}$ be a tower of field extensions with K Galois, such that $[K : K^+] = 2$ and $[K^+ : \mathbb{Q}] = g$. We define

$$T^{\mathrm{der}} = \mathrm{Ker}\left(\mathbf{R}_{K/\mathbb{Q}}(\mathbb{G}_m) \xrightarrow{N_{K/K^+}} \mathbf{R}_{K^+/\mathbb{Q}}(\mathbb{G}_m)\right) = \mathbf{R}_{K^+/\mathbb{Q}}\mathbf{R}^{(1)}_{K/K^+}(\mathbb{G}_m) \subset \mathrm{Sp}_{2g}\,,$$

and

$$T = \mathrm{Ker}\left(\mathbb{G}_m \times_{\mathrm{Spec}(\mathbb{Q})} \mathbf{R}_{K/\mathbb{Q}}(\mathbb{G}_m) \xrightarrow{(x,y)\mapsto x^{-1}N_{K/K^+}(y)} \mathbf{R}_{K^+/\mathbb{Q}}(\mathbb{G}_m)\right) \subset \mathrm{GSp}_{2g}.$$

They fit in the short exact sequence

$$1 \longrightarrow T^{\mathrm{der}} \longrightarrow T \longrightarrow \mathbb{G}_m \longrightarrow 1 \ .$$

Also, recall that we have been using τ_T (resp $\tau_{T^{\mathrm{der}}}$) to denote the Tamagawa numbers $\tau_{\mathbb{Q}}(T)$ and $\tau_{\mathbb{Q}}(T^{\mathrm{der}})$. We will show the following.

Proposition A.1. *One has* $\tau_{T^{\mathrm{der}}} = \tau_{K^+}(\mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m) = 2$.

For the case of T, the result varies with the extension.

Proposition A.2. *Assume that K is a Galois CM field and K^+ is its maximal totally real subfield. Then we have $\tau_T \leq 2$, and moreover :*

- *If g is odd then $\tau_T = 1$.*
- *If $K/\mathbb{Q}$ is cyclic, then $\tau_T = 1$.*
- *If $g = 2$ then $\tau_T = 1$ when $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ and $\tau_T = 2$ when $\mathrm{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$.*

More results and details will appear in the second author's forthcoming thesis.

We base our approach the following formula of Ono.

Theorem A.3 ([Ono63])**.** *Let $\mathbf{T}$ be an algebraic torus defined over a number field F and split over some Galois extension L. Then its Tamagawa number can be computed as*

$$\tau_F(\mathbf{T}) = \frac{|H^1(L/F, \mathbf{X}^\star(\mathbf{T}))|}{|Ш^1(\mathbf{T})|}.$$

Here $\mathbf{X}^\star(\mathbf{T})$ denotes the character lattice of $\mathbf{T}$. The symbol $Ш^1(\mathbf{T})$ denotes the corresponding Tate-Shafarevich group defined by

$$Ш^i(\mathbf{T}) = \mathrm{Ker}(H^i(L/F, \mathbf{T}) \to \prod_v H^i(L_w/F_v, \mathbf{T})), \tag{A.1}$$

where v runs over the primes of F and w is a prime of L with $w|v$.

Our approach is to do the computation on the level of character lattices. A very important consequence of Tate-Nakayama duality theorem (see [PR91, Theorem 6.10]) is that for a torus $\mathbf{T}$ as in the previous theorem, the Pontryagin dual of $Ш^1(\mathbf{T})$ is isomorphic to $Ш^2(\mathbf{X}^\star(\mathbf{T}))$, so it suffices to compute $|Ш^2(\mathbf{X}^\star(\mathbf{T}))|$.

The proof of proposition A.1 will be done in the next section. The proof of proposition A.2 occupies sections A.2 to A.5. In section A.6 we present an example not covered by proposition A.2. In section A.7 we present a computation for the numerator that illustrates the difficulties that arise for a general torus (not assuming that T is contructed from a single field).

A.1. **Computation of $\tau_{T^{\mathrm{der}}}$.** We write the proof of proposition A.1 using Theorem A.3. Since the Tamagawa number is preserved by restriction of scalars we have $\tau_{T^{\mathrm{der}}} = \tau_{\mathbb{Q}}(\mathbf{R}_{K^+/\mathbb{Q}}\mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m) = \tau_{K^+}(\mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m)$. The cohomology of the characters of a norm 1 torus is obtained by a classic computation that one can see for instance in the proof of the Hasse norm theorem in [PR91, Theorem 6.11]. We have

$$\hat{H}^i(K/\mathbb{Q}, \mathbf{X}^\star(T^{\mathrm{der}})) = \hat{H}^i(K/K^+, \mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m) = \hat{H}^{i+1}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \text{ if } i \text{ is odd} \\ \{0\} \text{ if } i \text{ is even} \end{cases} .$$

In particular, $|Ш^1(T^{\mathrm{der}})| = |Ш^2(\mathbf{X}^\star(T^{\mathrm{der}}))| \leq |H^2(\mathbf{X}^\star(T^{\mathrm{der}}))| = 1$. We conclude $\tau_{T^{\mathrm{der}}} = \frac{2}{1} = 2$.

This proves proposition A.1.

A.2. **Computation of the first cohomology group of the character lattice.** From now on, we focus on the proof of proposition A.2, and therefore we will assume that K is a CM-field with K^+ its maximal totally real subfield. Let ι be the nontrivial element of $\mathrm{Gal}(K/K^+)$, and let Γ, Γ^+ denote respectively the Galois groups of $K/\mathbb{Q}$, and $K^+/\mathbb{Q}$. Note that K^+ is indeed Galois over $\mathbb{Q}$ by virtue of K being a CM-field. The torus T arises as the subtorus of $\mathbf{R}_{K/\mathbb{Q}}(\mathbb{G}_m)$ with the set of $\mathbb{Q}$-points consisting of elements $x \in K^\times$ such that $x\iota(x) \in \mathbb{Q}$ and $T^{\mathrm{der}}(\mathbb{Q})$ is the set of elements $x \in K^\times$ such that $x\iota(x) = 1$. We have the following exact sequence of finite groups:

$$1 \longrightarrow \langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow \Gamma \longrightarrow \Gamma^+ \longrightarrow 1 \ . \tag{A.2}$$

For each $\sigma \in \Gamma^+$ fix a preimage $\hat{\sigma} \in \Gamma$. We get the description of $\mathbf{X}^\star(\mathbf{R}_{K/\mathbb{Q}}(\mathbb{G}_m)) = \mathbb{Z}[\Gamma]$ as the set of $\mathbb{Z}$-linear combinations of $\hat{\sigma}$ and $\hat{\sigma}\iota$ with $\sigma \in \Gamma^+$.

The embedding of T in $\mathbf{R}_{K/\mathbb{Q}}(\mathbb{G}_m)$ gives us a surjective map $\mathbf{X}^\star(\mathbf{R}_{K/\mathbb{Q}}(\mathbb{G}_m)) \to \mathbf{X}^\star(T)$. For $\chi = \sum_{\sigma\in\Gamma^+} a_\sigma \hat{\sigma} + \sum_{\sigma\in\Gamma^+} b_\sigma \hat{\sigma}\iota \in \mathbf{X}^\star(\mathbf{R}_{K/\mathbb{Q}}(\mathbb{G}_m))$ and $t \in T(\mathbb{Q})$, we have

$$\begin{aligned}
\chi(t) &= \prod_{\sigma\in\Gamma^+} \hat{\sigma}(t)^{a_\sigma} \prod_{\sigma\in\Gamma^+} \hat{\sigma}(\iota(t))^{b_\sigma} \\
&= \prod_{\sigma\in\Gamma^+} \hat{\sigma}(t)^{a_\sigma} \prod_{\sigma\in\Gamma^+} \hat{\sigma}\left(\lambda t^{-1}\right)^{b_\sigma} \qquad \text{where } t\iota(t) = \lambda \in \mathbb{Q}^\times \\
&= \lambda^{\sum_{\sigma\in\Gamma^+} b_\sigma} \prod_{\sigma\in\Gamma^+} \hat{\sigma}(t)^{a_\sigma - b_\sigma}.
\end{aligned}$$

We get the descriptions :

$$\mathbf{X}^\star(T) = \mathbb{Z}[\Gamma] / \left\{ \sum_{\sigma\in\Gamma^+} a_\sigma \hat{\sigma} + \sum_{\sigma\in\Gamma^+} b_\sigma \hat{\sigma}\iota : a_\sigma = b_\sigma \text{ for } \sigma \in \Gamma^+ \text{ and } \sum_{\sigma\in\Gamma^+} b_\sigma = 0 \right\} \tag{A.3}$$

$$= \mathbb{Z}[\Gamma]/L, \tag{A.4}$$

where $L = \{\sum_{\sigma\in\Gamma^+} a_\sigma \hat{\sigma}(1+\iota) : \sum_{\sigma\in\Gamma^+} a_\sigma = 0\}$. For T^{der}, we have $\lambda = 1$ so we recover

$$\mathbf{X}^\star(T^{\mathrm{der}}) = \mathbb{Z}[\Gamma]/\{x = \iota(x)\} = \mathbb{Z}[\widehat{\Gamma^+}] \otimes \mathbb{Z}[\iota]/\langle 1+\iota \rangle = \mathbf{X}^\star(\mathbf{R}_{K^+/\mathbb{Q}}\mathbf{R}^{(1)}_{K/K^+}(\mathbb{G}_m)). \tag{A.5}$$

In order to compute $H^1(K/\mathbb{Q}, \mathbf{X}^\star(T))$ we use the inflation-restriction exact sequence, which one can find in [GS06, Proposition 3.3.14 p.65]. To simplify notations, let $\Lambda = \mathbf{X}^\star(T) = \mathbb{Z}[\Gamma]/L$ as in (A.3).

The inflation-restriction exact sequence associated with the short exact sequence (A.2) takes the form

$$0 \longrightarrow H^1(\Gamma^+, \Lambda^{\mathbb{Z}/2\mathbb{Z}}) \longrightarrow H^1(\Gamma, \Lambda) \longrightarrow H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda)^{\Gamma^+} \longrightarrow H^2(\Gamma^+, \Lambda^{\mathbb{Z}/2\mathbb{Z}}) \longrightarrow H^2(\Gamma, \Lambda) \ . \tag{A.6}$$

Lemma A.4. *The sequence (A.6) can be rewritten as*

$$0 \longrightarrow 0 \longrightarrow H^1(\Gamma, \Lambda) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\frac{1+(-1)^g}{2}} \longrightarrow {\Gamma^+}^{\mathrm{ab}} \longrightarrow H^2(\Gamma, \Lambda) \ . \tag{A.7}$$

In particular, $\tau_T \leq |H^1(\Gamma, \mathbf{X}^\star(T))| \leq 2$.

Proof. Let $x \in \mathbb{Z}[\Gamma]$, and let $[x]$ denote its class in Λ. Clearly $[x]$ is fixed by ι if and only if $x - \iota x \in L$, and since every element of L is fixed by ι, then so must $x - \iota x$ which forces $x = \iota x$. Therefore,

$$\Lambda^{\mathbb{Z}/2\mathbb{Z}} = \{\sum_{\sigma\in\Gamma^+} a_\sigma \hat{\sigma}(1+\iota)\} / \left\langle \sum_{\sigma\in\Gamma^+} a_\sigma \hat{\sigma}(1+\iota) : \sum_{\sigma\in\Gamma^+} a_\sigma = 0 \right\rangle \cong \mathbb{Z}[\Gamma^+]/I,$$

where I is the augmentation ideal of $\mathbb{Z}[\Gamma^+]$, i.e. the subspace of sum-zero vectors. Further, observe that $\mathbb{Z}[\Gamma^+]/I \cong \mathbb{Z}$ as Γ^+-modules where $\mathbb{Z}$ has trivial Γ^+-action (by definition of I). We get that

$$H^1(\Gamma^+, \Lambda^{\mathbb{Z}/2\mathbb{Z}}) \cong H^1(\Gamma^+, \mathbb{Z}) = \mathrm{Hom}(\Gamma^+, \mathbb{Z}) = \{0\}.$$

Also, using the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0\ , \tag{A.8}$$

since the middle term is uniquely divisible hence cohomologically trivial, one has

$$H^2(\Gamma^+, \mathbb{Z}) \cong H^1(\Gamma^+, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\Gamma^+, \mathbb{Q}/\mathbb{Z}) \cong {\Gamma^+}^{ab}.$$

The only term left to compute is $H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda)^{\Gamma^+}$. As a $\mathbb{Z}/2\mathbb{Z}$-module, we can write $\Lambda = \mathbb{Z}^g \oplus \mathbb{Z}^g/L$ where $L = \{(a,a) : a = (a_1, \ldots, a_g)\ \sum a_i = 0\}$, and $\mathbb{Z}/2\mathbb{Z}$ acts as $(a,b) \mapsto (b,a)$. Therefore, we have

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^g \oplus \mathbb{Z}^g = \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]^g \longrightarrow \Lambda \longrightarrow 0\ .$$

Since the middle term is cohomologically trivial as a $\mathbb{Z}/2\mathbb{Z}$-module, and $\mathbb{Z}/2/Z$ acts trivially on L, we have

$$H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda) \cong H^2(\mathbb{Z}/2\mathbb{Z}, L) \cong \hat{H}^0(\mathbb{Z}/2\mathbb{Z}, L) \cong L/2L.$$

To compute $L/2L$, we view L as a submodule of $\mathbb{Z}^g$ of zero-sum elements. Recall that by construction of L as a group algebra, Γ^+ acts transitively on L.

Let $\mathbf{a} = (a_1, \ldots, a_g) \in L$. We want to compute $(L/2L)^{\Gamma^+}$, and for that we reason on the parity of a_i's.

- If all a_i are even, then $\mathbf{a} = 2\mathbf{a}'$ and $\mathbf{a}' \in L$, hence $\mathbf{a} \in 2L$.
- If $\mathbf{a}$ has a_i even and a_j odd, considering a permutation $\sigma \in \Gamma^+$ sending the ith coordinate to the jth, we have that the jth coordinate of $\mathbf{a} - \sigma\mathbf{a}$ is $a_j - a_i$ which is odd, hence $\mathbf{a} - \sigma\mathbf{a} \notin 2L$.
- The last case to consider is when all a_i are odd. In that case, $\sum_i a_i$ has the same parity as g, so $\mathbf{a} \in L$ can only happen if g is even. One can prove that every element of $L/2L$ has a representative of the form $\mathbf{a} = (a_1, \ldots, a_g) \in \mathbb{Z}^g$ with $\sum_i a_i = 0$ and $|a_i| \leq 1$. When g is even and all a_i are odd, the only possible such elements are vectors with half the coordinates being -1 and the other half 1. Moreover all such vectors are in the same coset of $2L$ (one can permute the ± 1 coordinates by adding ± 2).

This shows that $(L/2L)^{\Gamma^+}$ contains no nontrivial element when g is odd, and only one when g is even and conclude the proof. □

Corollary A.5. *When g is odd, one has $H^1(K/\mathbb{Q}, \mathbf{X}^\star(T)) = \{0\}$ and*

$$H^2(K/\mathbb{Q}, \mathbf{X}^\star(T)) \cong {\Gamma^+}^{\mathrm{ab}}.$$

In particular, $H^2(K/\mathbb{Q}, \mathbf{X}^\star(T))$ has odd order, and so does $Ш^1(T)$, since it is dual to $Ш^2(\mathbf{X}^\star(T)) \subset H^2(K/\mathbb{Q}, \mathbf{X}^\star(T))$.

Proof. The first equality comes directly from the sequence (A.7).

For the second equality, taking duals of the exact sequence (2.2), one gets

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbf{X}^\star(T) \longrightarrow \mathbf{X}^\star(T^{\mathrm{der}}) \longrightarrow 0 \ .$$

The cohomology of this sequence gives us

$$H^1(\Gamma, \mathbf{X}^\star(T)) \longrightarrow H^1(\Gamma, \mathbf{X}^\star(T^{\mathrm{der}})) \longrightarrow H^2(\Gamma, \mathbb{Z}) \longrightarrow H^2(\Gamma, \mathbf{X}^\star(T)) \longrightarrow H^2(\Gamma, \mathbf{X}^\star(T^{\mathrm{der}})) \ .$$

We computed the cohomology $H^i(\Gamma, \mathbf{X}^\star(T^{\mathrm{der}})) = H^i(\mathbb{Z}/2\mathbb{Z}, \mathbf{X}^\star(\mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m))$ in section A.1.

We can plug in in $H^1(\Gamma, \mathbf{X}^\star(T)) = \{0\} = H^2(\Gamma, \mathbf{X}^\star(T^{\mathrm{der}}))$, $H^2(\Gamma, \mathbb{Z}) \cong H^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \cong \Gamma^{ab}$, and $H^2(\Gamma, \mathbf{X}^\star(T^{\mathrm{der}})) \cong \mathbb{Z}/2\mathbb{Z}$, which gives us

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \Gamma^{ab} \longrightarrow H^2(\Gamma, \mathbf{X}^\star(T)) \longrightarrow 0 \ ,$$

as desired. □

Proposition A.6. *When the sequence (A.2) splits, one has* $H^1(K/\mathbb{Q}, \mathbf{X}^\star(T)) = \begin{cases} \{0\} \text{ if } g \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} \text{ if } g \text{ is even} \end{cases}$. *In particular, this gives an alternate proof of the triviality of* $H^1(K/\mathbb{Q}, \mathbf{X}^\star(T))$ *whenever* g *is odd.*

Proof. Since (A.2) splits, one can write the inflation-restriction exact sequence associated with the short exact sequence

$$1 \longrightarrow \Gamma^+ \longrightarrow \Gamma \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \ .$$

This gives us

$$0 \longrightarrow H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda^{\Gamma^+}) \longrightarrow H^1(\Gamma, \Lambda) \longrightarrow H^1(\Gamma^+, \Lambda)^{\mathbb{Z}/2\mathbb{Z}} \longrightarrow H^2(\mathbb{Z}/2\mathbb{Z}, \Lambda^{\Gamma^+}) \longrightarrow H^2(\Gamma, \Lambda) \ . \tag{A.9}$$

Since the sequence (A.2) splits, we have $\mathbb{Z}[\Gamma] = \mathbb{Z}[\Gamma^+] \otimes \mathbb{Z}[\iota]$ as a Γ-module. We have $\Lambda = \mathbb{Z}[\Gamma^+] \otimes \mathbb{Z}[\iota]/I \otimes (1+\iota)$ where I is the augmentation ideal of $\mathbb{Z}[\Gamma^+]$. Since $\mathbb{Z}[\Gamma^+]$ is an induced module, and $I \otimes (1+\iota) \cong I$ as a Γ^+-module, we get $\hat{H}^i(\Gamma^+, \Lambda) = \hat{H}^{i+1}(\Gamma^+, I)$. Now since $\mathbb{Z} = \mathbb{Z}[\Gamma^+]/I$ with $\mathbb{Z}$ seen as a trivial module, the same argument yields $\hat{H}^i(\Gamma^+, \Lambda) \cong \hat{H}^i(\Gamma^+, \mathbb{Z})$. In particular, $H^1(\Gamma^+, \Lambda) = \{0\}$ so the sequence (A.9) gives an isomorphism $H^1(\Gamma, \Lambda) \cong H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda^{\Gamma^+})$.

Direct computations using that $\overset{+}{\sigma} := \sum_{\sigma \in \Gamma^+} \sigma$ spans the set of Γ^+-fixed elements of $\mathbb{Z}[\Gamma^+]$ give us that $\{1 \otimes (1+\iota), \overset{+}{\sigma} \otimes \iota\}$ is a $\mathbb{Z}$-basis for Λ^{Γ^+}, on which ι acts via $\begin{pmatrix} 1 & g \\ 0 & -1 \end{pmatrix}$. Identifying the space with $\mathbb{Z}^2$ we can compute cocycles and coboundaries. Coboundaries are of the form $a_\iota = (-gb, 2b)$ for $b \in \mathbb{Z}$. Cocycles are of the form $a_\iota = (a, b)$ with $2a + gb = 0$. Thus, if b is even then it is a coboundary, if b is odd then g cannot be odd, and so we only get a nontrivial cocycle with g even and b odd. The difference of two nontrivial cocycles has an even second entry, so it is a coboundary. This proves $H^1(\mathbb{Z}/2\mathbb{Z}, \Lambda^{\Gamma^+}) = \begin{cases} \{0\} \text{ if } g \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} \text{ if } g \text{ is even} \end{cases}$ as desired.

For the last assertion in the proposition, when g is odd, by the Schur-Zassenhaus theorem (see [Rot95, Theorem 7.41]) the sequence (A.2) splits and we get our result immediately. □

A.3. **Case g odd.** Now we are ready to show

Lemma A.7. *If g is odd, then $\tau_T = \dfrac{1}{1} = 1$.*

Proof. Using corollary A.5 we have that $|H^1(K/\mathbb{Q}, \mathbf{X}^\star(T))| = 1$, and $Ш^1(T)$ has odd order. To show that $Ш^1(T)$ is trivial, it suffices to show it is 2-torsion. The cohomology of the sequence 2.2 yields

$$H^1(K/\mathbb{Q}, T^{\mathrm{der}}) \longrightarrow H^1(K/\mathbb{Q}, T) \longrightarrow H^1(K/\mathbb{Q}, \mathbb{G}_m) = 1 \ ,$$

where the right equality holds by Hilbert 90. So we have a surjection of $H^1(K/\mathbb{Q}, T^{\mathrm{der}})$ onto $H^1(K/\mathbb{Q}, T)$. We claim that $H^1(K/\mathbb{Q}, T)$ is 2-torsion; it suffices so show that $H^1(K/\mathbb{Q}, T^{\mathrm{der}})$ is.

Taking the cohomology of the following sequence where the middle term is cohomologically trivial,

$$1 \longrightarrow \mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m \longrightarrow \mathbf{R}_{K/K^+}\mathbb{G}_m \xrightarrow{N_{K/K^+}} \mathbb{G}_m \longrightarrow 1 \ ,$$

we have $H^1(K/K^+, \mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m) \cong \hat{H}^0(K/K^+, \mathbb{G}_m) = (K^+)^\times / N_{K/K^+}K^\times$.

This gives us

$$H^1(K/\mathbb{Q}, T^{\mathrm{der}}) = H^1(K/\mathbb{Q}, \mathbf{R}_{K/\mathbb{Q}}\mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m) = H^1(K/K^+, \mathbf{R}^{(1)}_{K/K^+}\mathbb{G}_m) \cong (K^+)^\times / N_{K/K^+}K^\times.$$

This group is 2-torsion, hence so is $H^1(K/\mathbb{Q}, T)$ and $Ш^1(T)$ is a subgroup of the latter. We can conclude that $Ш(T)$ is a 2-torsion group of odd order, hence it is trivial.

We can conclude using A.3 that $\tau_T = \frac{1}{1} = 1$.

Note that one need not use corollary A.5 to know that $Ш^1(T)$ has odd order and hence is trivial. Indeed, given that the extension K/K^+ is quadratic, by the Chebotarev density theorem, we know that there is a prime $\mathfrak{p} \in K^+$ inert in the extension K/K^+. Since $\mathfrak{p}$ is stable under ι, which is of order 2, then its decomposition group $\Gamma(\mathfrak{p})$ has even order, and therefore odd index in Γ. Now it suffices to look at the restriction-corestriction sequence $H^1(\Gamma, T) \to H^1(\Gamma(\mathfrak{p}), T) \to H^1(\Gamma, T)$. The composition of the two maps is just multiplication by $n = [\Gamma : \Gamma(\mathfrak{p})]$. By definition of $Ш^1(T)$, this subgroup of $H^1(\Gamma, T)$ is killed by the restriction map, hence it is n-torsion, and we know n is odd, as desired. □

A.4. **The case $K/\mathbb{Q}$ cyclic.**

Lemma A.8. *When K is cyclic, we have $\tau(T) = \frac{1}{1} = 1$. In particular, this holds when $K^+/\mathbb{Q}$ is cyclic of odd order.*

Proof. Write $\mathbf{X}^\star(T) = \mathbb{Z}[\Gamma]/L$ as in (A.3).

By virtue of K being cyclic, the Tate cohomology is 2-periodic so

$$H^1(K/\mathbb{Q}, \mathbf{X}^\star(T)) \cong H^2(K/\mathbb{Q}, L) \cong \hat{H}^0(K/\mathbb{Q}, L).$$

Since L has trivial ι action, we can see it as the augmentation ideal of $\mathbb{Z}[\Gamma^+]$, which has no Γ^+-fixed point, as any augmentation ideal. In particular it has no Γ-fixed point and so $\hat{H}^0(K/\mathbb{Q}, L) = \{0\}$.

Again using the fact that K is cyclic, we get that $Ш^1(T)$ is trivial. Indeed, by the Chebotarev density theorem, every cyclic extension has a prime $p \in \mathbb{Z}$ that will stay inert, and therefore $\Gamma(p) = \Gamma$ where $\Gamma(p)$ is the corresponding decomposition group. Therefore, the map in the definition of $Ш^1(T)$ is injective.

We can conclude by Theorem A.3 that $\tau_T = \frac{1}{1} = 1$. □

A.5. **Case $g = 2$.**

Lemma A.9. *When $g = 2$ we have $\tau_T = 1$ if Γ is cyclic, otherwise $\tau_T = 2$.*

Proof. The first case is a consequence of Lemma A.8. If Γ isn't cyclic, the only possibility is $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^2$.

In that case, proposition A.6 gives $H^1(\Gamma, \mathbf{X}^\star(T)) \cong \mathbb{Z}/2\mathbb{Z}$. Concerning the Tate-Shafarevich group, is was shown by Cortella in [Cor97] that $Ш^1(T) = \{0\}$ if $g < 4$.

Alternatively, since Γ is abelian, every proper cyclic subgroup appears as decomposition group. In this specific case, it is a consequence of the Chinese remainder theorem and quadratic reciprocity. Using SAGE computations (see below), we obtain that the map

$$H^2(\Gamma, \mathbf{X}^\star(T)) \longrightarrow H^2(\mathbb{Z}/2\mathbb{Z} \times \{0\}, \mathbf{X}^\star(T)) \oplus H^2(\{0\} \times \mathbb{Z}/2\mathbb{Z}, \mathbf{X}^\star(T))$$

is injective, therefore $Ш^1(T) = 0$. □

A.6. **Computing the Tamagawa numbers with SAGE.** The second author implemented methods in SAGE to deal with algebraic tori through their character lattices. Those methods should eventually be added to SAGE in a future release.

Here we briefly describe the computation of the Tamagawa number which arises in Example 6.4.2, where $K = \mathbb{Q}[T]/f(T)$ for

$$f(T) = T^3 - 6T^7 + 13T^6 - 10T^5 + T^4 - 30T^3 + 117T^2 - 162T + 81.$$

We have $\Gamma = \mathrm{Gal}(K/\mathbb{Q}) = \mathbb{Z}/4 \oplus \langle \iota \rangle$ where ι denotes the complex involution.

Nakayama duality lets us compute the Tamagawa number as a function of the character lattice,

$$\tau_{\mathbb{Q}}(T) = \frac{|H^1(\mathbb{Q}, \mathbf{X}^\star(T))|}{|Ш^2(\mathbf{X}^\star(T))|}.$$

Let Λ denote $\mathbf{X}^\star(T)$. We build Λ in SAGE by inducing the trivial lattice $\mathbb{Z} = \mathbf{X}^\star(\mathbb{G}_m)$ to Γ, build the sublattice of zero sum elements of ι-fixed points, and quotient the former by the latter.

We can compute the first cohomology group by computing cocycles as solutions of linear equations in $\Lambda^{|\Gamma|}$. However for this example, Proposition A.6 gives us $H^1(\Gamma, \Lambda) = 2$. For the denominator, we build a method that given $\mathcal{H}$ a collection of subgroups of Γ, and a Γ-lattice L, computes

$$Ш^1_{\mathcal{H}}(L) = \mathrm{Ker}\left(H^1(\Gamma, L) \to \bigoplus_{\Delta \in \mathcal{H}} H^1(\Delta, L) \right),$$

by checking what cocycles restrict to coboundaries on all $\Delta \in \mathcal{H}$.

Consider the embedding $\varphi : \Lambda \to \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \Lambda$ (with Γ-action on the left component) via $a \mapsto \sum_{g \in \Gamma} g \otimes g^{-1}a$. We build

$$\Lambda' = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \Lambda / \varphi(\Lambda).$$

Since $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \Lambda$ is induced, it is cohomologically trivial, hence $\hat{H}^i(\Gamma, \Lambda) = \hat{H}^{i-1}(\Gamma, \Lambda)$ for all $i \in \mathbb{Z}$. Consequently we have

$$Ш^1_{\mathcal{H}}(\Lambda') \supset Ш^1(\Lambda') = Ш^2(\Lambda).$$

We take $\mathcal{H}$ to be the list of cyclic subgroups of Γ. They all arise as decomposition groups since Γ is cyclic. Using the program, we get $|Ш^1_{\mathcal{H}}(\Lambda')| = 1 \geq Ш^2(\Lambda) \geq 1$, hence

$$\tau(T) = \frac{|H^1(\Gamma, \Lambda)|}{|Ш^2(\Lambda)|} = \frac{2}{1} = 2.$$

A.7. **The numerator in Ono's formula: general case.** We give some indications for the general case in which T is described by a CM algebra $K = \bigoplus_{i=1}^t K_i$, each K_i being a CM field. The Rosati involution on K is still denoted as ι, with fixed subalgebra $K^+ = \bigoplus_{i=1}^t K_i^+$. In this section we denote by Γ the absolute Galois group of $\mathbb{Q}$. Our modest aim is to understand $\left|H^1(\Gamma, \mathbf{X}^\star(T))\right|$ through Kottwitz's isomorphism (see [Kot84a] (2.4.1) and §2.4.3):

$$H^1(\Gamma, \mathbf{X}^\star(T)) \cong \pi_0(\hat{T}^\Gamma),$$

where $\hat{T}$ is the dual $\mathbb{C}$-torus. This isomorphism is valid for all tori.

To describe $\mathbf{X}^\star(T)$, we first write T as

$$T = (\mathbb{G}_m \times T^{\mathrm{der}}) / \{(z, z) : z \in \mu_2\}, \quad \mu_2 := \{\pm 1\}.$$

Choose a subset $\Phi = \bigsqcup_{i=1}^t \Phi_i$ of $\mathrm{Hom}_{\mathbb{Q}\text{-alg}}(K, \overline{\mathbb{Q}})$, such that $\Phi_i \subset \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(K_i, \overline{\mathbb{Q}})$ and

$$\mathrm{Hom}_{\mathbb{Q}\text{-alg}}(K_i, \overline{\mathbb{Q}}) = \Phi_i \sqcup \Phi_i \iota$$

for all $i = 1, \ldots, t$. Note that $|\Phi| = g$. It is well-known that

$$\mathbf{X}^\star(T^{\mathrm{der}}) = \bigoplus_{\phi \in \Phi} \mathbb{Z}\epsilon_\phi$$

for some basis $\{\epsilon_\phi\}_{\phi \in \Phi}$. Then Γ permutes $\{\pm\epsilon_\phi\}_{\phi \in \Phi}$ by

$$\sigma\epsilon_\phi = \begin{cases} \epsilon_\psi, & \text{if } \sigma\phi = \psi \in \Phi \\ -\epsilon_\psi, & \text{if } \sigma\phi = \psi\iota \in \Phi\iota, \end{cases} \quad \phi, \psi \in \Phi. \tag{A.10}$$

The inclusion $\mu_2 \hookrightarrow T^{\mathrm{der}}$ corresponds to the map

$$\begin{aligned} \mathbf{X}^\star(T^{\mathrm{der}}) &\twoheadrightarrow \mathbf{X}^\star(\mu_2) = \mathbb{Z}/2\mathbb{Z} \\ \sum_{\phi \in \Phi} x_\phi \epsilon_\phi &\mapsto \sum_{\phi \in \Phi} x_\phi \mod 2. \end{aligned}$$

Write $\mathbf{X}^\star(\mathbb{G}_m) = \mathbb{Z}\eta$, where η is the standard generator. Applying Cartier duality to the exact sequence

$$1 \to \mu_2 \to \mathbb{G}_m \times T^{\mathrm{der}} \to T \to 1,$$

we obtain

$$\mathbf{X}^\star(T) = \left\{ t\eta + \sum_{\phi \in \Phi} x_\phi \epsilon_\phi : t + \sum_\phi x_\phi \in 2\mathbb{Z} \right\} \subset \mathbf{X}^\star(\mathbb{G}_m) \oplus \mathbf{X}^\star(T^{\mathrm{der}}),$$

$$\text{a basis of } \mathbf{X}^\star(T): \quad \{2\eta\} \sqcup \{\eta + \epsilon_\phi : \phi \in \Phi\}.$$

The element 2η is surely Γ-invariant. On the other hand, for $\phi, \psi \in \Phi$ and $\sigma \in \Gamma$, we derive from (A.10) that

$$\sigma(\eta + \epsilon_\phi) = \begin{cases} \eta + \epsilon_\phi, & \text{if } \sigma\phi = \psi \in \Phi \\ 2\eta - (\eta + \epsilon_\psi), & \text{if } \sigma\phi = \psi\iota \in \Phi\iota. \end{cases} \tag{A.11}$$

The Γ-action on $\hat{T} := \mathbf{X}^\star(T) \otimes \mathbb{C}^\times \cong \mathbb{C}^\times \times (\mathbb{C}^\times)^\Phi$ is thus

$$\sigma \cdot (z, \overbrace{1, \ldots, \underbrace{w}_{\phi}, \ldots, 1}^{\Phi}) = \begin{cases} (z, 1, \ldots, \underbrace{w}_{\psi}, \ldots, 1), & \text{if } \sigma\phi = \psi \in \Phi \\ (zw, 1, \ldots, \underbrace{w^{-1}}_{\psi}, \ldots, 1), & \text{if } \sigma\phi = \psi\iota \in \Phi\iota. \end{cases}$$

Recall from the description of $\mathbf{X}^\star(T^{\mathrm{der}})$ that $\widehat{T^{\mathrm{der}}}$ can be identified with $(\mathbb{C}^\times)^\Phi$. Note that $(\widehat{T^{\mathrm{der}}})^\Gamma$ is finite since T^{der} is anisotropic.

Lemma A.10. *There is a canonical isomorphism $(\widehat{T^{\mathrm{der}}})^\Gamma \cong \mu_2^t$ characterized as follows. For $(a_i)_{i=1}^t \in \mu_2^t$, the corresponding $\hat{t} = (\hat{t}_\phi)_{\phi\in\Phi} \in (\widehat{T^{\mathrm{der}}})^\Gamma$ is specified by*

$$\forall 1 \le i \le t, \quad \phi \in \Phi_i \implies \hat{t}_\phi = a_i.$$

Proof. Shapiro's lemma reduces the computation of $(\widehat{T^{\mathrm{der}}})^\Gamma$ or $H^1(\Gamma, \mathbf{X}^\star(T))$ to the easy case $t = 1$ over the base field K^+. □

By dualizing $1 \to T^{\mathrm{der}} \to T \to \mathbb{G}_m \to 1$ into $1 \to \mathbb{C}^\times \to \hat{T} \to \widehat{T_0} \to 1$, then taking Γ-invariants, we obtain the exact sequence

$$1 \to \mathbb{C}^\times \to \hat{T}^\Gamma \to (\widehat{T^{\mathrm{der}}})^\Gamma.$$

It induces

$$\pi_0(\hat{T}^\Gamma) = \hat{T}^\Gamma/\mathbb{C}^\times \hookrightarrow (\widehat{T^{\mathrm{der}}})^\Gamma = \pi_0((\widehat{T^{\mathrm{der}}})^\Gamma).$$

For each $\sigma \in \Gamma$, set $\Phi(\sigma) := \{\phi \in \Phi : \sigma\phi \notin \Phi\}$. For each i, set $\Phi_i(\sigma) := \Phi(\sigma) \cap \Phi_i$.

Proposition A.11. *For any $(a_i)_i \in \mu_2^t$, the corresponding element $\hat{t} \in (\widehat{T^{\mathrm{der}}})^\Gamma$ belongs to the image of $\hat{T}^\Gamma/\mathbb{C}^\times$ if and only if*

$$A(a_1, \ldots, a_t; \sigma) := \sum_{\substack{1 \le i \le t \\ a_i = -1}} |\Phi_i(\sigma)| \in 2\mathbb{Z}$$

for all $\sigma \in \Gamma_F$.

Proof. Identify $\widehat{T^{\mathrm{der}}}$ with $(\mathbb{C}^\times)^\Phi$. Identify $\hat{T}$ with $\mathbb{C}^\times \times (\mathbb{C}^\times)^\Phi$ using the basis $\{2\eta\} \sqcup \{\eta + \epsilon_\phi : \phi \in \Phi\}$ of $\mathbf{X}^\star(T)$; the homomorphism $\hat{T} \to \widehat{T^{\mathrm{der}}}$ is simply the projection.

Note that $\hat{t}$ is the image of $(1, \hat{t}) \in \hat{T}$. It comes from $\hat{T}^\Gamma/\mathbb{C}^\times$ if and only if $(1, \hat{t})$ (or any other preimage) is Γ-invariant. For all $\sigma \in \Gamma$, Lemma A.10 and the description (A.11) lead to

$$\sigma \cdot (1, \hat{t}) = \left((-1)^{A(a_1, \ldots, a_t; \sigma)}, \hat{t}\right).$$

The assertion follows at once. □

To illustrate the use of proposition A.11, we prove the following

Proposition A.12. *If $t = 1$ and g is odd, then $H^1(\Gamma, \mathbf{X}^\star(T)) \cong \pi_0(\hat{T}^\Gamma)$ is trivial.*

Proof. It suffices to show $A(-1, \ldots, -1; c) = |\Phi(c)| \notin 2\mathbb{Z}$ where $c \in \Gamma$ is the complex conjugation. Indeed, $c\phi = \phi\iota$ for all $\phi \in \Phi$ by generalities on CM fields, hence $\Phi(c) = \Phi$ has g elements, which is odd. □

Note that K is not assumed to be Galois over $\mathbb{Q}$.

Kottwitz's theory also relates Tate–Shafarevich groups to similar objects attached to dual tori; see §4 of [Kot84a]. Nevertheless, we are not yet able to determine the Tate–Shafarevich group of T by this approach in the non-Galois case.

References

[AG17] Jeffrey D. Achter and Julia Gordon, *Elliptic curves, random matrices and orbital integrals*, Pacific J. Math. **286** (2017), no. 1, 1–24, With an appendix by S. Ali Altuğ. MR 3582398

[AW13] Jeffrey D. Achter and Siman Wong, *Quotients of elliptic curves over finite fields*, Int. J. Number Theory **9** (2013), no. 6, 1395–1412. MR 3103894

[AW15] Jeffrey Achter and Cassandra Williams, *Local heuristics and an exact formula for abelian surfaces over finite fields*, Canad. Math. Bull. **58** (2015), no. 4, 673–691. MR 3415659

[Bit11] Rony A. Bitan, *The discriminant of an algebraic torus*, J. Number Theory **131** (2011), no. 9, 1657–1671. MR 2802140 (2012g:11118)

[Bou85] N. Bourbaki, *Eléménts de mathématique. algèbre commutative*, Masson, 1985.

[Clo90] Laurent Clozel, *The fundamental lemma for stable base change*, Duke Math. J. **61** (1990), no. 1, 255–302.

[Con14] Brian Conrad, *Reductive group schemes*, Autour des schémas en groupes. Vol. I, Panor. Synthèses, vol. 42/43, Soc. Math. France, Paris, 2014, pp. 93–444. MR 3362641

[Cor97] Anne Cortella, *The Hasse principle for the similarities of bilinear forms*, Algebra i Analiz **9** (1997), no. 4, 98–118. MR 1604012

[FLN10] Edward Frenkel, Robert Langlands, and Báo Châu Ngô, *Formule des traces et fonctorialité: le début d'un programme*, Ann. Sci. Math. Québec **34** (2010), no. 2, 199–243. MR 2779866 (2012c:11240)

[Gek03] Ernst-Ulrich Gekeler, *Frobenius distributions of elliptic curves over finite prime fields*, Int. Math. Res. Not. (2003), no. 37, 1999–2018.

[Gor19] Julia Gordon, *Orbital integrals and normalization of measures*, In preparation; to appear in proceedings of "On the Langlands Program: Endoscopy and Beyond", NUS.

[Gro97] Benedict H. Gross, *On the motive of a reductive group*, Invent. Math. **130** (1997), 287–313.

[Gro98] Benedict H. Gross, *On the Satake isomorphism*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, Cambridge, 1998, pp. 223–237. MR 1696481

[GS06] Philippe Gille and Tamás Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006. MR 2266528

[GW19] Jonathan Gerhard and Cassandra Williams, *Local heuristics and an exact formula for abelian varieties of odd prime dimension over finite fields*, New York J. Math. **25** (2019), 123–144. MR 3904880

[GY00] Wee Tack Gan and Jiu-Kang Yu, *Group schemes and local densities*, Duke Math. Journal **105** (2000), no. 3, 497–524.

[Kis17] Mark Kisin, *mod p points on Shimura varieties of abelian type*, J. Amer. Math. Soc. **30** (2017), no. 3, 819–914. MR 3630089

[Kot82] Robert E. Kottwitz, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49** (1982), no. 4, 785–806. MR 683003 (84k:20020)

[Kot84a] Robert Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math Journal **51** (1984), no. 3, 611–650.

[Kot84b] Robert E. Kottwitz, *Shimura varieties and twisted orbital integrals*, Math. Ann. **269** (1984), no. 3, 287–300. MR 761308

[Kot90] ———, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 161–209. MR 1044820

[Kot92] ———, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444. MR 1124982 (93a:11053)

[Kot05] ———, *Harmonic analysis on reductive p-adic groups and Lie algebras*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 393–522. MR 2192014 (2006m:22016)

[Lan73] R. P. Langlands, *Modular forms and ℓ-adic representations*, 361–500. Lecture Notes in Math., Vol. 349. MR 0354617

[Oes82] Joseph Oesterlé, *Réduction modulo p^n des sous-ensembles analytiques fermés de $\mathbf{Z}_p^N$*, Invent. Math. **66** (1982), no. 2, 325–341. MR 656627

[Ono63] Takashi Ono, *On the Tamagawa number of algebraic tori*, Ann. of Math. (2) **78** (1963), 47–73. MR 0156851

[PR91] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Academic Press, Inc., San Diego, 1991.

[Rot95] Joseph J. Rotman, *An introduction to the theory of groups*, fourth ed., Graduate Texts in Mathematics, vol. 148, Springer-Verlag, New York, 1995. MR 1307623

[Ser81] Jean-Pierre Serre, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 54, 323–401.

[Shy77] Jih Min Shyr, *On some class number relations of algebraic tori*, Michigan Math. J. **24** (1977), no. 3, 365–377. MR 0491596

[Tat66] John Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144. MR MR0206004 (34 #5829)

[Vey92] Willem Veys, *Reduction modulo p^n of p-adic subanalytic sets*, Math. Proc. Cambridge Philos. Soc. **112** (1992), no. 3, 483–486. MR 1177996 (93i:11142)

[Vos98] V. E. Voskresenskiĭ, *Algebraic groups and their birational invariants*, Translations of Mathematical Monographs, vol. 179, American Mathematical Society, Providence, RI, 1998, Translated from the Russian manuscript by Boris Kunyavski [Boris È. Kunyavskiĭ]. MR 1634406

[Wat69] William C. Waterhouse, *Abelian varieties over finite fields*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 521–560. MR 0265369

[Wei82] André Weil, *Adeles and algebraic groups*, Progress in mathematics, vol. 23, Birkhäuser, 1982.

COLORADO STATE UNIVERSITY, FORT COLLINS, CO

Email address: `achter@math.colostate.edu`

BOSTON UNIVERSITY, BOSTON, MA

Email address: `saaltug@bu.edu`

UNIVERSITY OF TORONTO, TORONTO, ON, CANADA

Email address: `lgarcia@math.toronto.edu`

UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BC

Email address: `gor@math.ubc.ca`