Hilbert-Siegel moduli spaces in positive characteristic

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Hilbert-Siegel varieties are moduli spaces for abelian varieties equipped with an action by an order O_K in a fixed, totally real field K. As such, they include both the Siegel moduli spaces (use $K = \mathbb{Q}$ and the action is the standard one) and Hilbert-Blumenthal varieties (where the dimension of K is the same as that of the abelian varieties in question). In this paper we study certain phenomena associated to Hilbert-Siegel varieties in positive characteristic. Specifically, we show that ordinary points are dense in moduli spaces of mildly inseparably polarized abelian varieties with action by a given totally real field. Moreover, we introduce a combinatorial invariant of the first cohomology of an abelian variety which allows us to compute and explain the singularities of such a moduli space.

The problem considered here arises in two distinct but closely related lines of inquiry. On one hand, recall that if X is an abelian variety over a field k of characteristic p, then its p-torsion is described by $X[p](\bar{k}) \cong (\mathbb{Z}/p\mathbb{Z})^\rho$ for some ρ. This integer ρ, the p-rank, is between zero and dim X. When ρ is maximal, the abelian variety is said to be ordinary. Deuring shows that the generic elliptic curve is ordinary [4]. Mumford announces [12], and Norman and Oort prove [14], the obvious generalization of this statement to higher dimension: ordinary points are dense in the moduli space of polarized abelian varieties. Wedhorn has recently obtained similar results [18] for families of principally polarized abelian varieties with given ring of endomorphisms.

On the other hand, moduli spaces of PEL type – those parametrizing abelian varieties with certain polarization, endomorphism and level-structure data – are important spaces in their own right. Roughly speaking, when the characteristic of the ground field is relatively prime to the moduli problem, the resulting space is smooth. When the characteristic resonates with the moduli functor, things get interesting and the spaces get singular. The singularities of such spaces have attracted considerable attention. In particular, the program established in [17] studies singularities arising from ramification of the endomorphism ring or the level structure at p. The present work is complementary to these efforts, in that it addresses bad-reduction behavior coming from inseparable polarizations. (The final paragraph of this paper observes that a weak form of theorem 3.2.2 proves a special case of a conjecture of [17].)

The first section gives the precise definition of the moduli stacks in question. The second section collects a number of results on the deformation theory of polarized abelian varieties equipped with an action by a ring. Since it requires little extra notation and no extra work, we state and prove many of these results in a slightly broader context than that required for Hilbert-Siegel varieties. We conclude by applying these techniques to deduce the density of the ordinary locus in certain cases and to compute the structure of local rings on these spaces.

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1 Moduli spaces

The moduli spaces under consideration are defined in the following way. Let B a finite-dimensional $\mathbb{Q}$-algebra with positive involution $*$, and let $O_B \subset B$ be an order stable under $*$ and maximal at a rational prime p.

Let O_B be an order in a finite-dimensional $\mathbb{Q}$-algebra B with positive involution $*$. Let Δ be the product of all primes which ramify in B. Finally, let g and d be natural numbers.

Definition 1.1   We denote by $\widetilde{\mathcal{A}}_{g,d}^{O_B}$ the category of triples $(X/S, \iota, \lambda)$ where

  i. $X \to S \to \operatorname{Spec} \mathbb{Z}[\frac{1}{\Delta}]$ is an abelian scheme of relative dimension g.

 ii. $O_B \overset{\iota}{\hookrightarrow} \operatorname{End}(X)$ is a ring homomorphism taking 1 to id_X, so that $\operatorname{Lie}(X)$ is a locally free $O_B \otimes O_S$-module.

iii. $X \overset{\lambda}{\to} X^\vee$ is a polarization of degree d^2, taking the given involution on O_B to the Rosati involution of $\operatorname{End}(X) \otimes \mathbb{Q}$.

Fix an algebraically closed field $\mathbb{Z}[\frac{1}{\Delta}] \to k$ of characteristic $p > 0$. Denote the reduction of the global moduli space modulo p by

$$\mathcal{A}_{g,d}^{O_B} \overset{\text{def}}{=} \widetilde{\mathcal{A}}_{g,d}^{O_B} \times_{\operatorname{Spec} \mathbb{Z}[\frac{1}{\Delta}]} \operatorname{Spec} k.$$

We will often use the locution "polarized O_B-abelian variety" to denote a point $(X/k, \iota, \lambda) \in \mathcal{A}_{g,d}^{O_B}(k)$.

Remark 1.2   The demanded compatibilities in (ii) and (iii) are quite reasonable requests of our moduli space. The freeness constraint in (ii) expresses one instance of Kottwitz's "determinantal condition" [8]. While modifying this condition still yields a reasonable moduli space, other such conditions may forbid the existence of ordinary points. Moreover, if d is invertible on S and the center of B is fixed by $*$, then (ii) is automatic. (If d is not invertible, however, then $\operatorname{Lie}(X)$ is *not* necessarily locally free over $O_B \otimes O_S$; (ii) is not a vacuous condition.)

It may be worth making (iii)'s meaning explicit, too. An ample line bundle $\mathcal{L}$ on an abelian variety X over a field k induces an isogeny $\phi_{\mathcal{L}} : X \to X^\vee \overset{\text{def}}{=} \operatorname{Pic}^0(X)$, $x \mapsto \mathcal{L} \otimes T_x^* \mathcal{L}^{-1}$. An isogeny arising in this way is called a polarization of X/k. If X is an abelian scheme over S, then a polarization of X is a map $\lambda : X \to X^\vee$ which is a polarization of abelian varieties at every geometric point of S. The degree of a polarization is simply its degree as an isogeny, that is, the rank of its kernel. Any polarization induces a Rosati involution on $\operatorname{End}(X) \otimes \mathbb{Q}$, defined by $\alpha^\dagger = \lambda^{-1} \circ \alpha^\vee \circ \lambda$. We insist that, for any $b \in O_B$, $\iota(b^*) = \iota(b)^\dagger$.

The functor $(X/S, \iota, \lambda) \mapsto S$ clearly reveals $\widetilde{\mathcal{A}}_{g,d}^{O_B}$ as a fibered category over $\operatorname{Sch}_{\mathbb{Z}[\frac{1}{\Delta}]}$.

Theorem 1.3 The category $\widetilde{\mathcal{A}}_{g,d}^{O_B}$ is an algebraic stack over $\mathbb{Z}[\frac{1}{\Delta}]$ in the sense of [3].

Proof The sketch in Théorème 1.20 of [16], which treats the case where O_B is a totally real field of dimension g, is an exegesis of Artin's method which works for general B. Alternatively, one can consider the forgetful functor $\widetilde{\mathcal{A}}_{g,d}^{O_B} \xrightarrow{\phi} \widetilde{\mathcal{A}}_{g,d}$ to the moduli space of abelian schemes of relative dimension g equipped with a polarization of degree d^2. The target space is well-known to be an algebraic stack, and it suffices to show that ϕ is relatively representable. This last property, in turn, comes down to the algebraicity of condition (ii) above. Since the freeness of $\mathrm{Lie}(X/k)$ is a statement about the mutiplicities of various representations $\chi : O_B \to \mathrm{Aut}\,\mathrm{Lie}(X)$, and since these multiplicities are clearly constructible functions, ϕ is representable. (In fact, a class-number argument shows that ϕ is quasifinite, and a rigidity statement on homomorphisms shows ϕ is even finite; but we will not need this here.)$\heartsuit$

2   Deformation theory

For the most part, we will attempt to understand $\mathcal{A}_{g,d}^{O_B}$ by studying its local behavior. Indeed, let $\mathrm{Art}_p(k)$ be the category of Artin local k-algebras with residue field k and let $(X/k, \iota, \lambda) \in \mathcal{A}_{g,d}^{O_B}(k)$ be any k-point. Then $\mathrm{Def}(X)$ is the covariant functor $\mathrm{Art}_p(k) \to \mathrm{Set}$ taking R to the set of all pairs $(\widetilde{X}/R, \phi)$, where $\widetilde{X}/R$ is an abelian scheme and $\widetilde{X} \times_{\mathrm{Spec}\,R} \mathrm{Spec}\,k \xrightarrow{\phi} X$ is an isomorphism. The closed subfunctors $\mathrm{Def}(X, \lambda)$ and $\mathrm{Def}(X, \iota, \lambda)$ are defined analogously. Tautologically, $\widehat{O}_{\mathcal{A}_{g,d}^{O_B}, (X, \iota, \lambda)}$ pro-represents $\mathrm{Def}(X, \iota, \lambda)$. We describe here three different approaches to the deformation theory of a polarized O_B-abelian variety.

As a preliminary step we remark that $\mathrm{Def}(X/k, \iota, \lambda)$ depends not on the global arithmetic of B but on its p-adic behavior. To make this precise, let $X[p^\infty] \overset{\mathrm{def}}{=} \lim_{\substack{\longrightarrow \\ n}} X[p^n]$ be the p-divisible (or Barsotti-Tate) group of X. Denote by $\widehat{\iota}$ an action of $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ on a p-divisible group.

Lemma 2.1 The natural map $\mathrm{Def}(X, \iota, \lambda) \to \mathrm{Def}(X[p^\infty], \widehat{\iota}, \lambda)$ is an isomorphism of functors.

Proof The Serre-Tate theorem (first announced in section 6 of [9], although the reader might profitably consult V.2.3 of [11] or 1.2.1 of [7]) implies that $\mathrm{Def}(X, \iota, \lambda) \to \mathrm{Def}(X[p^\infty], \iota, \lambda)$ is an isomorphism. The only novelty here lies in passing from an O_B-action ι to an $O_B \otimes \mathbb{Z}_p$-action $\widehat{\iota}$. However, any p-divisible group has a canonical structure of $\mathbb{Z}_p$-module. Thus, one may indifferently place an O_B- or $O_B \otimes \mathbb{Z}_p$-structure on $X[p^\infty]$.$\heartsuit$

Remark 2.2 Since B is by hypothesis unramified and maximal at p, $O_B \otimes \mathbb{Z}_p \cong \oplus_j \mathrm{Mat}_{s_j} W(\mathbb{F}_{p^{f_j}})$. Idempotents of $O_B \otimes \mathbb{Z}_p$ will often give a (noncanonical) decomposition $(X[p^\infty], \lambda) = \oplus_j (X[p^\infty]_j, \lambda_j)$, where each $(X[p^\infty]_j, \lambda_j)$ is a polarized p-divisible group equipped with an action by a certain ring $W(\mathbb{F}_{p^j})$. In order to show that (X, ι, λ) admits a deformation to an ordinary abelian variety, it thus suffices to produce an ordinary deformation of each triple $(X[p^\infty]_j, W(\mathbb{F}_{p^j}) \xrightarrow{\iota_j} \mathrm{End}(X[p^\infty]_j), \lambda_j)$. Thus, if desired, 2.1 would often let us assume that $O_B \otimes \mathbb{Z}_p \cong W(\mathbb{F}_{p^f})$, i.e., that B has the same local structure as a number field inert at p.

2.1 Dieudonné theory

Reducing a question of abelian varieties to one of formal groups would not be much of an improvement, were it not for Dieudonné theory. There are several different functors which come under the rubric of Dieudonné theory. They all associate to a formal or p-divisible group a σ-linear algebraic object. We will make heavy use of the covariant Dieudonné theory, efficiently documented in [19], which has the distinct advantage of working over an arbitrary base ring.

Let G be a formal p-divisible group over a ring R of characteristic p. Associated to it is its Dieudonné module $\mathbb{D}_*(G) \stackrel{\text{def}}{=} \operatorname{Hom}(\widehat{W}, G) \subset G(R[\![T]\!])$, the group of p-typical curves on G. This group is a module over the local Cartier ring $\operatorname{Cart}_p(R) \stackrel{\text{def}}{=} (\operatorname{End} \widehat{W})^{\text{opp}}$ (see [19] 4.17). When R is a perfect field k, the local Cartier ring is

$$\operatorname{Cart}_p(k) = \frac{W(k)[F][\![V]\!]}{(FV - p, VaF - a^\tau, Fa - a^\sigma F, Va^\sigma - aV)}$$

where σ and τ are the Frobenius and Verschiebung of $W(k)$. For general R there is always an embedding $W(R) \hookrightarrow \operatorname{Cart}_p(R)$, and σ- and τ-linear elements F and V, respectively, of $\operatorname{Cart}_p(R)$.

A Dieudonné module over R is a V-adically separated and complete $\operatorname{Cart}_p(R)$-module M such that $V : M \to M$ is injective and M/VM is a locally free, finite R-module. Over a perfect field k, a Dieudonné module may then be thought of as a free $W(k)$-module of rank height(G), equipped with σ- and τ-linear operators F and V satisfying certain identities; and M/VM is canonically the tangent space of G. The fundamental theorem of Dieudonné theory says that there is an equivalence between the category of formal groups over R and the category of Dieudonné modules over R ([19] 4.23).

For simplicity of exposition, let X/k be an abelian variety with p-rank zero. By the Dieudonné module M of X we mean $\mathbb{D}_*(X[p^\infty])$. It is a free, rank $2g = 2 \dim X$ module over $W(k)$. By functoriality, a polarization $X \stackrel{\lambda}{\to} X^\vee$ induces a homomorphism of Dieudonné modules $\mathbb{D}_*(X) \stackrel{\lambda_*}{\to} \mathbb{D}_*(X^\vee)$. Moreover $\mathbb{D}_*(X^\vee)$ is, up to Tate twist, the $W(k)$-linear dual of $\mathbb{D}_*(X)$. Carefully following through dualities (as in [12] or [13]; see also section 5.1 of [1]) shows λ induces a $W(k)$-linear pairing $\langle \cdot, \cdot \rangle : M \times M \to W(k)$ such that $\langle m, m' \rangle = -\langle m', m \rangle$ and $\langle Fm, m' \rangle = \langle m, Vm' \rangle^\sigma$. This is the Dieudonné-theoretic analogue of the Riemann form of a polarized complex abelian variety.

Roughly speaking, twisting the action of F on M by a nilpotent endomorphism gives a family of deformations of X. A special case of this idea is made precise in the following lemma.

Construction 2.1.1 Let (X, ι, λ) be a polarized O_B-abelian variety, and let M be its Dieudonné module. To any nilpotent endomorphism $\nu : M \to M$ corresponds a deformation $\widetilde{M}/k[\![\epsilon]\!]$. If $\langle \cdot, \cdot \rangle$ extends to $\widetilde{M}$, this construction gives a deformation $(\widetilde{X}/k[\![\epsilon]\!], \lambda)$. If ν commutes with O_B, this construction gives a deformation $(\widetilde{X}/k[\![\epsilon]\!], \iota, \lambda)$.

Proof This is the main theorem of section 1 of [13], although our coordinate-free formulation more closely follows section 4 of [2]. Let ϵ be the Teichmüller lift of $\underline{\epsilon}$ to $W(k[\![\underline{\epsilon}]\!])$. Set $\mu = \operatorname{id} + \epsilon\nu : \widetilde{M} \to \widetilde{M}$, where $\widetilde{M} = M \otimes_{\operatorname{Cart}_p(k)} \operatorname{Cart}_p(k[\![\underline{\epsilon}]\!])$. Set $\widetilde{F} = \mu \circ F$, a twisted form of the original Frobenius. One can adapt the Verschiebung as well so that $(\widetilde{M}, \widetilde{F})$ is a Dieudonné module. Now recall the Serre-Tate theorem (lemma 2.1) to get the full statement.♡

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Suppose X is equipped with an action by O_K, where K is a totally real field inert at p. The action of O_K on M becomes particularly easy to describe. There is a canonical structure of $W(k)$-module on M, and O_K acts on M via embeddings $O_K \hookrightarrow W(k)$. For convenience's sake, identify $\mathrm{Hom}(O_K, W(k))$ with $\mathbb{Z}/f\mathbb{Z}$ by fixing one such map $O_K \otimes \mathbb{Z}_p \cong W(\mathbb{F}_{p^f}) \xrightarrow{\sigma_0} W(k)$. Let σ be the Frobenius of $W(\mathbb{F}_{p^f})$ and set $\sigma_i = \sigma_0 \circ \sigma^i : W(\mathbb{F}_{p^f}) \hookrightarrow W(k)$ for $1 \leq i \leq f-1$. Let M^i be the eigenspace where O_K acts via σ_i. There is a decomposition of M as a $W(k)$-module,

$$M = \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} M^i.$$

This is *not* a direct sum of Dieudonné modules. Indeed, the Frobenius and Verschiebung operators interweave the M^i. For any element m^i of M^i,

$$
\begin{aligned}
F(\alpha m^i) &= F(\sigma_i(\alpha)m^i) \\
&= \sigma_i(\alpha)^\sigma F m^i \\
&= \sigma_{i+1}(\alpha)F m^i.
\end{aligned}
$$

Thus, $FM^i \subseteq M^{i+1}$ with the expected arithmetic for indices. Similarly, $VM^i \subseteq VM^{i-1}$.

Let $\langle \cdot, \cdot \rangle$ be the nondegenerate alternating form on M induced by the polarization λ. It turns out that $\langle \cdot, \cdot \rangle|_{M^i}$ is nondegenerate for each i. Recall that, as K is totally real, the Rosati involution is actually trivial on elements of K. For any $\alpha \in O_K$ and $m^i \in M^i, m^j \in M^j$, on one hand $\langle \alpha m^i, m^j \rangle = \langle \sigma_i(\alpha)m^i, m^j \rangle = \sigma_i(\alpha)\langle m^i, m^j \rangle$; on the other, $\langle \alpha m^i, m^j \rangle = \langle m^i, \alpha m^j \rangle = \langle m^i, \sigma_j(\alpha)m^j \rangle = \sigma_j(\alpha)\langle m^i, m^j \rangle$. If $i \neq j$, then choosing any α with $\sigma_i(\alpha) \neq \sigma_j(\alpha)$ shows that $\langle m^i, m^j \rangle = 0$.

It is possible – and, for the explicit deformation theory which follows, desirable – to choose bases for M which clearly expose the behavior of F, V and $\langle \cdot, \cdot \rangle$.

Lemma 2.1.2 Let $(M, \iota, \langle \cdot, \cdot \rangle)$ be a quasipolarized Dieudonné module equipped with an action by O_K. Let M^i be the eigenspace corresponding to σ_i as above. There are $W(k)$-bases $\{e_1^i, \cdots, e_{2r}^i\}$ and $\{f_1^i, \cdots, f_{2r}^i\}$ for M^i such that

- $Fe_j^i \in \{f_j^{i+1}, pf_j^{i+1}\}$.

- If $W(k)(Fe_{r+j}^i)$ is a direct summand then so is $W(k)(Fe_j^i)$.

- $\langle e_j^i, e_{j'}^{i'} \rangle \neq 0 \iff i = i', |j - j'| = r$.

- $\langle e_j^i, e_{r+j}^i \rangle = p^{\delta_j^i}$ for some $\delta_j^i \in \mathbb{Z}_{\geq 0}$.

Notationally, let $f_l^i = \sum_{j=1}^r a_{jl}^i e_j^i$. Then (a_{jl}^i) is an automorphism of $\langle \cdot, \cdot \rangle|_{M^i}$, and in particular is invertible over $W(k)$.

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Proof In view of the previous computations, the proof is essentially a careful meditation on the elementary divisors lemma. It may be worth pointing out that, in the absence of an O_K-action, this recovers the "displayed module" of [13].

Fix $i \in \mathbb{Z}/f\mathbb{Z}$. By the freeness hypothesis, $(M/FM)^i \cong \mathrm{Lie}(X^\vee)^i \cong k^r$. By, say, the elementary divisors lemma there are bases e^i_{1j} and f^i_{1j} so that $Fe^i_{1j} \in \{f^i_{1j}, pf^i_{1j}\}$. Let's agree to say that F acts unimodularly, or with index zero, on e^i_{1j} if $W(k)Fe^i_{1j}$ is a direct summand; and with index one if $Fe^i_{1j} = pf^i_{1j}$. (In general, the index of an element $x \in M$ is the largest $n \in \mathbb{Z}$ with $x \in p^n M$; and if T is a $[\sigma^{\pm 1}$-]linear operator on M, declare that T acts with index $(\mathrm{ind}\, Tx - \mathrm{ind}\, x)$ on x.) The idea is simply to diagonalize this basis with respect to the symplectic form.

Order the e^i_{1j} and f^i_{1j} so that $\mathrm{ord}_p\langle e^i_{11}, e^i_{1,r+1}\rangle$ is minimal among all $\mathrm{ord}_p\langle e^i_{1j}, e^i_{1l}\rangle$. We may further choose these first elements so that $\mathrm{ind}\, Fe^i_{11} + \mathrm{ind}\, Fe^i_{1,r+1}$ is *maximal* over all i, j with $\mathrm{ord}_p\langle e^i_{1j}, e^i_{1j}\rangle$ minimal. Start orthogonalizing, by setting

$$
e^i_{2j} = \begin{cases} e^i_{1j} & j = 1, r+1 \\ e^i_{1j} + \frac{\langle e^i_{1j}, e^i_{1,r+1}\rangle}{\langle e^i_{11}, e^i_{1,r+1}\rangle} e^i_{11} + \frac{\langle e^i_{1j}, e^i_{11}\rangle}{\langle e^i_{11}, e^i_{1,r+1}\rangle} e^i_{1,r+1} & j \neq 1, r+1 \end{cases}.
$$

Note that, by the minimality of $\mathrm{ord}_p\langle e^i_{11}, e^i_{1,r+1}\rangle$, the division is permissible over $W(k)$. Moreover, for $j \neq 1, r+1$, $\langle e^i_{2j}, e^i_{21}\rangle = \langle e^i_{2j}, e^i_{2,r+1}\rangle = 0$. Indeed,

$$
\begin{aligned}
\langle e^i_{21}, e^i_{2j}\rangle &= \langle e^i_{11}, e^i_{1j}\rangle + \frac{\langle e^i_{1j}, e^i_{11}\rangle}{\langle e^i_{11}, e^i_{1,r+1}\rangle}\langle e^i_{11}, e^i_{1,r+1}\rangle \\
&= 0
\end{aligned}
$$

and the same computation works for $\langle e^i_{2j}, e^i_{2,r+1}\rangle$.

The only issue is whether $e^i_{21}, \cdots, e^i_{2,2r}$ is still a good basis for describing the action of F. Fix a $j \neq 1, r+1$.

If Fe^i_{1j} is unimodular, then Fe^i_{2j} is clearly such, too. If Fe^i_{1j} has index one, however, there is still a small amount of verification to be done. Ideally, Fe^i_{2j} should have index one as well.

If $\mathrm{ord}_p\langle e^i_{1j}, e^i_{11}\rangle$ and $\mathrm{ord}_p\langle e^i_{1j}, e^i_{1,r+1}\rangle$ are strictly greater than $\mathrm{ord}_p\langle e^i_{11}, e^i_{1,r+1}\rangle$, then there's no problem, as $e^i_{2j} - e^i_{1j} \in pM$.

The situation to worry about is the following: $Fe^i_{1j} = pf^i_{1j}$, $Fe^i_{11} = f^i_{11}$, $\mathrm{ord}_p\langle e^i_{1j}, e^i_{1,r+1}\rangle = \mathrm{ord}_p\langle e^i_{11}, e^i_{1,r+1}\rangle$. But then $\mathrm{ind}\, Fe^i_{1j} + \mathrm{ind}\, Fe^i_{1,r+1} > \mathrm{ind}\, Fe^i_{11} + \mathrm{ind}\, Fe^i_{1,r+1}$, contradicting the second assumption on e^i_{11} and $e^i_{1,r+1}$.

Now iterate this procedure, ultimately constructing $e^i_j = e^i_{rj}$, to finish; and the f^i_j are determined by the e^i_j. $\heartsuit$

Call any such choice of bases a normal form for (M, ι, λ_*). Empirically, it is *much* easier to write down deformations of Dieudonné modules which enjoy the following property:

(*) There is a normal form such that, for each $i \in \mathbb{Z}/f\mathbb{Z}$ and $1 \leq j \leq r$,
$Fe^i_j = f^{i+1}_j$ and $Fe^i_{r+j} = pf^{i+1}_{r+j}$.

Suppose such exists. Define certain direct summands of M in terms of the given normal form.

$$
\begin{array}{rclcrcl}
Q^i & = & \oplus_{j=1}^r W(k)e_j^i & \quad Q & = & \oplus_{i\in\mathbb{Z}/f\mathbb{Z}} Q^i \\
P^i & = & \oplus_{j=1}^r W(k)e_{r+j}^i & \quad P & = & \oplus_{i\in\mathbb{Z}/f\mathbb{Z}} P^i
\end{array}
$$

By the definition of normal form, these summands satisfy the following conditions:

  i. $M^i = P^i \oplus Q^i$, hence $M = P \oplus Q$.

  ii. $\langle P, P \rangle = \langle Q, Q \rangle = (0) \subset W(k)$.

  iii. $P \bmod pM = VM/pM$.

In fact, such summands characterize the sort of Dieudonné module we're after:

Lemma 2.1.3 M satisfies (*) if and only if there are $P^i, Q^i \subseteq M$ satisfying (i) through (iii).

Proof If M satisfies (*), then the P^i and Q^i obviously satisfy (i) through (iii), by the definition of normal form.

Conversely, suppose we are given such P^i and Q^i. Start with arbitrary $W(k)$-bases for P^i and Q^i, and diagonalize as in the proof of lemma 2.1.2. Since $\langle P, P \rangle = \langle Q, Q \rangle = 0$, the algorithm will merely produce new bases for P and Q. By (iii), F acts with index one on P. So it must act with index zero on Q, and (*) is satisfied.♡

Definition 2.1.4 Call a polarized abelian variety whose quasipolarized Dieudonné module satisfies (*) nice. With a slight abuse of nomenclature, say the $W(k)$-summand M^i is nice if there are P^i and Q^i as above. With a somewhat more serious abuse, we will sometimes say X and M are nice if the polarization is clear from context.

In view of the commutative diagram associated to any such decomposition,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & M = P \oplus Q & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & VM/pM & \longrightarrow & M/pM & \longrightarrow & M/VM & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & H^1(X, \mathcal{O}_X)^\vee & \longrightarrow & H_1^{\mathrm{dR}}(X) & \longrightarrow & \mathrm{Lie}(X) & \longrightarrow & 0
\end{array}
$$

this condition may be reasonably paraphrased as demanding that the Hodge filtration admit an isotropic lifting.

There is a nice rank $n(X) = (n_0, \cdots, n_{f-1})$ which measures the defect of (X, ι, λ) from nice. Set

$$n_i = \max \#\{j | 1 \leq j \leq r, Fe_j^i = f_j^{i+1}, Fe_{r+j}^i = pf_{r+j}^{i+1}\},$$

where the maximum is taken over all possible normal forms for M. Note that X is nice if and only if each $n_i = r$.

Remark 2.1.5 If X is separably polarized, then $\mathbb{D}_*(X)$ is nice. For suppose not. Then there are i and j so that F acts with index zero on e_j^i and e_{r+j}^i. On one hand, $\langle Fe_j^i, Fe_{r+j}^i \rangle = \langle e_j^i, VFe_{r+j}^i \rangle^\sigma = p$; on the other hand, $\langle Fe_j^i, Fe_{r+j}^i \rangle = \langle e_j^{i+1}, e_{r+j}^{i+1} \rangle$. Thus, $p | d$, contradicting the hypothesis of separability. $\heartsuit$

Remark 2.1.6 The following remarks, while not logically necessary in the sequel, may help give the reader some idea of the lay of the land.

i. One might reasonably ask whether it is necessary to consider *all* possible normal forms for M to determine its nice rank; it is certainly distasteful. In the special case where all elementary divisors of M are 1 or p, it suffices to consider a single normal form. Indeed, suppose M is such and that there is some normal form which is not visibly nice. Then there are $i \in \mathbb{Z}/f\mathbb{Z}$ and $1 \leq j \leq r$ with $\langle e_j^i, e_{r+j}^i \rangle = 1$, $F\{e_j^i, e_{r+j}^i\} = \{f_j^i, f_{r+j}^i\}$. Define P^i and Q^i as above, although $\langle P^i, Q^i \rangle \supsetneq (0)$. Any apparent improvement to the nice rank must come from finding $x, y \in pP^i + Q^i$, not both in Q^i, so that $\langle e_j^i + y, e_{r+j}^i + x \rangle = 0$, i.e., $\langle x, y \rangle = 1$. Given the constraints on the elementary divisors, this is impossible.

A similar argument shows the same claim when the relative dimension r is two. Unfortunately, it fails for arbitrary polarized O_K-abelian varieties; this may help explain why we only use this notion in studying mildly inseparable polarizations.

ii. In contrast with the p-rank, the nice rank depends on the integral structure of (X, ι, λ); it is not preserved by isogenies. Still, this rank induces a reasonable stratification on $\mathcal{A}_{g,d}^{O_K}$. Nice is an open condition; we sketch a proof. Suppose (X, ι, λ) is nice. This is equivalent to the existence of $W(k)$-submodules $Q, R \subset M$ so that $\langle Q, Q \rangle = \langle R, R \rangle = (0)$; $\dim_k FQ \bmod pM = \mathrm{rk}_{W(k)} Q = g$; and $\dim_k VR \bmod pM = \mathrm{rk}_{W(k)} R = g$ [simply take $R = V^{-1}P$]. Consider any deformation of this polarized Dieudonné module. Working only with the polarization, there are submodules $\widetilde{Q}, \widetilde{R}$, lifting Q and R, so that $\langle \widetilde{Q}, \widetilde{Q} \rangle = \langle \widetilde{R}, \widetilde{R} \rangle = (0)$. Since "having full rank under $F \bmod p$ or $V \bmod p$" is an open condition, the generic lifts $\widetilde{Q}$ and $\widetilde{R}$ still have $\dim_k F\widetilde{Q} \bmod pM = \dim_k V\widetilde{R} \bmod pM = g$. Now, the deformation of M also changes the action of $\widetilde{F}$ and $\widetilde{V}$; but if $F\widetilde{Q} \bmod pM$ has full rank, then so must the generic $\widetilde{F}\widetilde{Q} \bmod pM$. This argument works for any suitable summands, not just ones of full rank. Thus, if $\mathbb{N}^f$ is equipped with the product partial order, then the function $(X, \iota, \lambda) \mapsto n(X)$ is a lower semicontinuous function on $\mathcal{A}_{g,d}^{O_K}$. In fact, we will see in 3.2 that, in certain cases, the nice stratification recovers the stratfication by singularity.

Remark 2.1.7 It is not hard to adapt these notions to handle totally imaginary rings of endomorphisms, too. However, insofar as the smooth case is adequately addressed in [18], and the ramified case seems to require different techniques, we forgo the immediate temptation to generalize.

2.2 Crystalline Cohomology

Crystalline cohomology is another tool well-adapted to the study of p-divisible groups and abelian varieties in positive characteristic. The classical theory is ably documented in chapters IV and V of [6] and I of [1].

To an abelian scheme $X \xrightarrow{\pi} S = \operatorname{Spec} k$ we associate its Dieudonné crystal $\mathbb{D}^*(X) \stackrel{\text{def}}{=} H^1_{\text{cris}}(X) \stackrel{\text{def}}{=} R^1\pi_{*,\text{cris}}O_X$, a sheaf of crystals on S_{cris}. The Hodge filtration on $H^1_{\text{dR}}(X) \cong H^1_{\text{cris}}(X)(k)$ extends to a filtration of the actual Dieudonné crystal.

It is easy to formulate the crystalline solution to our deformation problem.

Lemma 2.2.1 Let S' be a divided power extension of S, and let (X, ι, λ) be a polarized O_B-abelian variety. To give a deformation of $(X/S, \iota, \lambda)$ to S' is to give $\operatorname{Fil}(S') \subset \mathbb{D}^*(X)(S')$ so that

  i. $\operatorname{Fil}(S')$ is a locally direct summand of $\mathbb{D}^*(X)(S')$ as a $O_{S'} \otimes O_B$-module.

  ii. $\operatorname{Fil}(S') \subset \mathbb{D}^*(X)(S')$ lifts the Hodge filtration.

  iii. $\operatorname{Fil}(S')$ is isotropic with respect to the induced bilinear form $\langle \cdot, \cdot \rangle_\lambda$.

Proof This is essentially the Grothendieck-Messing theory of admissible filtrations (see, e.g., V.4 of [6]). The elucidation of the O_B-structure is clear in view of 2.1.♡

We will often prefer to work with the linear dual $H_1^{\text{cris}}(X)$ and its (dual) Hodge filtration, as this exposes the connection between the crystalline theory and the covariant Dieudonné theory. Indeed, let $M = \mathbb{D}_*(X[p^\infty])$. There are canonical isomorphisms $M/pM = H_1^{\text{dR}}(X) = H_1^{\text{cris}}(X)(k)$ and $M/VM \cong \operatorname{Lie}(X)$ (see [1]). Dualizing the Hodge filtration yields

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Lie}(X^\vee)^\vee & \longrightarrow & H_1^{\text{dR}}(X) & \longrightarrow & \operatorname{Lie}(X) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \operatorname{Fil}(M/pM) = VM/pM & \longrightarrow & M/pM & \longrightarrow & M/VM & \longrightarrow & 0
\end{array}
$$

Let $M^* = \mathbb{D}_*(X^\vee)$; up to a Tate twist, it is the $W(k)$-linear dual of the free $W(k)$-module M. Clearly, $\operatorname{Fil}(M^*/pM^*)$ may be computed in the same way. Alternately, observe that $\operatorname{Fil}(M^*/pM^*) = \operatorname{Lie}(X^{\vee\vee})^\vee = \operatorname{Hom}(M/VM, k) = \{e^* \in M^* : (VM, e^*) = (0)\}$.

2.3 Kodaira-Spencer theory

A third approach, which actually works well in any characteristic, is Kodaira-Spencer theory. We refer to [15] for a careful exposition of the algebraic formulation of this technique, essentially due to Mumford and Grothendieck. Instead, we content ourselves with recalling that $\operatorname{Def}(X) \cong \widehat{\operatorname{Symm}}^\bullet(T_eX^\vee \otimes_k T_eX)$, and that the obstruction to lifting a polarization lives in $H^2(X, O_X) \cong H^1(X, O_X) \wedge_k H^1(X, O_X)$.

A modest variant of these classical results lets us study $\operatorname{Def}(X, \iota, \lambda)$. The first-order deformations are now parametrized by $T_eX \otimes_{k \otimes O_B} T_eX^\vee$, and the obstruction to lifting a polarization lives in $H^1(X, O_X) \wedge_{k \otimes O_B} H^1(X, O_X)$.

Theorem 2.3.1 Let $s = \dim_k T_eX \otimes_{k \otimes O_B} T_eX^\vee$ and $c = \dim_k(T_eX^\vee \wedge_{k \otimes O_B} T_eX^\vee)$. Then there are power series $a_1, \cdots, a_c$ so that

$$\text{Def}(X, \iota, \lambda) \cong \frac{k[\![t_1, \cdots, t_s]\!]}{(a_1, \cdots, a_c)}.$$

Proof The proof is quite similar to that of theorem 2.3.3 of [15], which proves the analogous result for $\text{Def}(X, \lambda)$. Clearly, $\text{Def}(X, \iota)$ is a smooth, pro-representable subfunctor of $\text{Def}(X)$. Moreover, either using general arguments from Kodaira-Spencer theory or (the dual of) the Dieudonné-theoretic description of $\text{Def}(X)$ in [13], we see that $\text{Def}(X, \iota)(k[\epsilon]/(\epsilon^2)) \cong \text{Hom}_{k \otimes O_B}((T_eX)^\vee, T_eX^\vee) = T_eX \otimes_{k \otimes O_B} T_eX^\vee$, and

$$\begin{aligned}
\text{Def}(X, \iota) &\cong \widehat{\text{Symm}}_k^\bullet(T_eX \otimes_{k \otimes O_B} T_eX^\vee) \\
&\cong k[\![t_1, \cdots, t_s]\!] \stackrel{\text{def}}{=} \mathcal{D}
\end{aligned}$$

It remains to compute the closed subfunctor $\text{Def}(X, \iota, \lambda)$ of $\text{Def}(X, \iota)$, necessarily represented by $\mathcal{D}/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{D}$. By the Artin-Rees lemma, there is an $n > 0$ so that $\mathfrak{m}^n \cap \mathfrak{a} = \mathfrak{m}^n \cap \mathfrak{m}\mathfrak{a}$, and thus

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\mathfrak{a}}{\mathfrak{m}\mathfrak{a}} & \longrightarrow & \dfrac{\mathcal{D}}{\mathfrak{m}\mathfrak{a} + \mathfrak{m}^n} & \longrightarrow & \dfrac{\mathcal{D}}{\mathfrak{a} + \mathfrak{m}^n} & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R' & \longrightarrow & 0
\end{array}
$$

is exact. Note that I is an ideal with $I^2 = \mathfrak{m}I = (0)$. The canonical surjections

$$
\begin{array}{ccc}
\mathcal{D} & \longrightarrow & \mathcal{D}/\mathfrak{a} \\
\downarrow & & \downarrow \\
R & \longrightarrow & R'
\end{array}
$$

give an O_B-abelian scheme $(X/R, \iota)$, a polarized O_B-abelian scheme $(X'/R', \iota', \lambda')$, and an isomorphism $(X, \iota) \otimes_R R' \cong (X', \iota')$. We work with the first de Rham homology $H_1^{\text{dR}}(X)$ and $H_1^{\text{dR}}(X')$ in order to apply the discussion of endomorphisms in section 2.1. The former is a free filtered R-module $\text{Fil}(X) = (T_eX^\vee)^\vee \subset H_1^{\text{dR}}(X)$ equipped with an eigenspace decomposition $\text{Fil}(X) = \oplus \text{Fil}^i(X)$, $H_1^{\text{dR}}(X) = \oplus H_1^{\text{dR}}(X)^i$, and $H_1^{\text{dR}}(X')$ is an analogous object over R'. Of course, $\oplus \text{Fil}^i(X) \subset \oplus H_1^{\text{dR}}(X)^i$ lifts $\oplus \text{Fil}^i(X') \subset H_1^{\text{dR}}(X')^i$. The polarization λ' induces a bilinear form on $H_1^{\text{dR}}(X')$, for which $\text{Fil}(X')$ is a Lagrangian subspace. The polarization lifts to X/R if and only if $\text{Fil}(X)$ is isotropic under the induced form. Choose a basis $\{b_j^i\}$ for each eigenspace $H_1^{\text{dR}}(X)^i$, and set

$$\widetilde{b}^i_{jl} = \langle b^i_j, b^i_l \rangle$$

if the center of B, acting on Fil^i, is fixed by the involution of B, and $\widetilde{b}^i_{jl} = \langle b^i_j, b^{\bar{i}}_l \rangle$ otherwise.

Since $\langle \mathrm{Fil}(X'), \mathrm{Fil}(X') \rangle = (0) \subset R'$, $\widetilde{b}^i_{jl} \in I$. For each i, j, l let b^i_{jl} be a lift of $\widetilde{b}^i_{jl}$ to $\mathfrak{a}$; these are the $a_1, \cdots, a_c$ promised in the theorem. Finally, let $\mathfrak{b} \subset \mathcal{D}$ be the ideal generated by all b^i_{jl}.

In fact, $\mathfrak{b} = \mathfrak{a}$. To see this, let $R'' = \frac{\mathcal{D}}{\mathfrak{b}+\mathfrak{m}\mathfrak{a}+\mathfrak{m}^n}$. By the construction of R'', $\langle \mathrm{Fil}(X''), \mathrm{Fil}(X'') \rangle = (0) \subset R''$. Consequently the polarization λ' lifts to R'' and there is a map $\mathcal{D}/\mathfrak{a} \to R''$. Thus, $\mathfrak{a} \subset \mathfrak{b} + \mathfrak{m}\mathfrak{a} + \mathfrak{m}^n$, and $\mathfrak{b} = \mathfrak{a}$.♡

Note that this gives a quick lower bound on the dimension of each component of $\mathcal{A}^{O_B}_{g,d}$. The following simple observation will be of decisive importance later.

Corollary 2.3.2 Let K be a totally real field of dimension $f = [K : \mathbb{Q}]$, and let $r = \frac{g}{f}$. Then the dimension of each component of $\mathcal{A}^{O_K}_{g,d}$ is at least $f \cdot \frac{r(r+1)}{2}$.

Proof   Indeed, $\dim_{O_K \otimes k} T_e X = \dim_{O_K \otimes k} T_e X^\vee = r$, so $s = \dim_k (T_e X \otimes_{O_K \otimes k} T_e X^\vee) = fr^2$; and $c = \dim_k (T_e X^\vee \wedge_{k \otimes O_K} T_e X^\vee) = f\frac{r(r-1)}{2}$. This gives a lower bound of $s - c = f\frac{r(r+1)}{2}$ on the dimension of the local ring at any point of $\mathcal{A}^{O_K}_{g,d}$, and thus on the dimension of each component.♡

Corollary 2.3.3 Let Z be an irreducible component of $\mathcal{A}^{O_K}_{g,d}$ containing an ordinary point. Then every point of Z has a lift to a ring of characteristic zero; $\dim_k Z = f\frac{r(r+1)}{2}$; and Z is (everywhere) a local complete intersection.

Proof   Observe that, as the Serre-Tate theory produces a lift of an ordinary abelian variety along with *all* its endomorphisms, any ordinary point of Z lifts. Now, by hypothesis, every point of Z is a specialization of an ordinary abelian scheme, and thus of a liftable point. Since liftability is a closed property – indeed, the "liftable locus" of a scheme $\mathcal{M}/W(k)$ is precisely the intersection of $\mathcal{M} \times \mathrm{Spec}\, k$ and the closure of $\mathcal{M} \times \mathrm{Spec}(\mathrm{Frac}\, W(k))$ inside $\mathcal{M}$ – the first claim follows.

As for the second claim, this is the dimension predicted from characteristic zero; but as every point lifts, the prediction must come true.

The third claim is now an immediate consequence of the second claim and the description of formal neighborhoods in 2.3.1.♡

3 Density of the ordinary locus

Armed with the deformation theory of section two, it makes sense to approach the objects defined in 1.1. Subsection 3.1 studies the deformation theory of the nice points defined in 2.1.4, and immediately deduces the density of the ordinary locus in the smooth case. The final subsection introduces singularities by allowing inseparability in the polarization. In cases of mild inseparability we compute the structure of the local rings, and again conclude that ordinary points are dense.

3.1 The ordinary locus of smooth moduli spaces

The condition introduced in 2.1.4 is a convenient hypothesis for the following result.

Lemma 3.1.1 Let K be a totally real field. Suppose $(X, \iota, \lambda) \in \mathcal{A}_{g,d}^{O_K}(k)$ is nice but not ordinary. Then (X, ι, λ) admits an infinitesimal deformation to a polarized O_K-abelian variety with strictly bigger p-rank.

Proof We use the covariant Dieudonné theory described in 2.1.1. The Serre-Tate theory assures us we may work directly with the p-divisible group $X[p^\infty] = X[p^\infty]' \oplus X[p^\infty]^{\text{tor}} \oplus X[p^\infty]^{\text{ét}}$, where $X[p^\infty]'$ is the local-local part of $X[p^\infty]$ which keeps X from being ordinary. By, say, the classification of p-divisible groups [10], this decomposition is stable under the O_K-action, so we may study $X[p^\infty]'$ and its Dieudonné module M. As explained in remark 2.2, we may and do assume that K is actually inert at p.

We will produce a nontrivial deformation of $X[p^\infty]$ to a family of p-divisible groups over $k[\![\epsilon]\!]$. The quasipolarization will be preserved; by the Serre-Tate theory, this gives a polarized formal abelian scheme $(\widetilde{X}/k[\![\epsilon]\!], \widetilde{\iota}, \widetilde{\lambda})$. By [EGA] III.5.4.5, this algebraizes to an honest abelian scheme.

Choose a normal basis for M as in lemma 2.1.2. Define a nilpotent endomorphism ν of M by

$$\nu(e_j^i) = \begin{cases} 0 & 1 \leq j \leq r \\ e_{j-r}^i & r+1 \leq j \leq 2r \end{cases}$$

and apply construction 2.1.1. As ν preserves the blocks M^i, and $\langle \mu(x), \mu(y) \rangle = \langle x, y \rangle$ for all $x, y \in M = M \otimes 1 \subset \widetilde{M}$, ι and λ extend to $\widetilde{M}$. Hence, this gives a deformation $(\widetilde{X}, \widetilde{\iota}, \widetilde{\lambda})$ of (X, ι, λ). It is worth remarking that it is exactly the nice condition which makes it so easy to produce deformations which preserve the quasipolarization.

In order to show that the p-rank has increased under our deformation, it's certainly enough to produce some $x \in \widetilde{M}$ and $l \in \mathbb{N}$ with $\widetilde{F}^l x = \gamma x$, $\gamma \in W(k(\!(\epsilon)\!))^\times$. (This is, of course, the same as showing that $\widetilde{F}$ is not nilpotent on $\widetilde{M}/\widetilde{V}\widetilde{M}$.) It is in fact slightly more convenient, and for the purposes of computing the p-rank harmless, to verify this for a geometric generic point of the formal deformation. So let $\widetilde{k} = k(\!(\epsilon)\!)^{\text{perf}}$, and base change to $\widetilde{k}$.

Consider, say, e_1^1; F acts unimodularly on it, so

$$
\begin{aligned}
\widetilde{F}e_1^1 &= \mu(f_1^2) \\
&= \sum_{j=1}^{r} a_{j1}^2 e_j^2 + \sum_{j=r+1}^{2r} (e_{j1}^2 + \epsilon e_{j-r,1}^2) \\
&= \sum_{j=1}^{r} (a_{j1}^2 + \epsilon a_{r+j,1}^2) e_j^2 + \sum_{j=r+1}^{2r} a_{j1}^2 e_j^2.
\end{aligned}
$$

Now, there is some $1 \leq j \leq r$ such that $a_{j1}^2 + \epsilon a_{r+j,1}^2 \in W(\widetilde{k})^\times$; otherwise, $p|a_{j1}^2$ for *all* j, and (a_{jk}^2) would be singular. So $\widetilde{F}e_1^1 \in W(\widetilde{k})^\times e_j^2 \oplus W(\widetilde{k})e_{l\,l\neq j}^2$, and $\widetilde{F}$ acts unimodularly on e_j^2. Continuing in this way, and remembering that there are only finitely many e_j^i, we produce some e_j^i on which $\widetilde{F}$ does not act nilpotently. ♡

3.2 Mildly inseparable polarizations

When a low power of p divides d, the moduli spaces tend to be singular but not unmanageably so. We consider here a class of such spaces.

As always, let K be a totally real field unramified at p of degree $f = [K : \mathbb{Q}]$. Throughout this section assume $d = p^f m$ with m prime to p. Moreover, since we make vital use of crystalline techniques, *assume throughout this section that $p > 3$.* Crystalline cohomology supplies a good description of the local geometry of $\mathcal{A}_{g,d}^{O_K}$. We start by computing an infinitesimal neighborhood of $(X/k, \iota, \lambda)$ in $\mathcal{A}_{g,d}^{O_K}$.

We have seen in 2.2.1 that to deform X/k to a PD extension R of k is to give an admissible filtration of $H_1^{\mathrm{cris}}(X)(R)$. An action ι extends to $\widetilde{X}$ if the filtration is O_K-linear, in the sense that

$$\mathrm{Fil}(X)(R) = \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} \mathrm{Fil}(X)(R)^i \subset \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} H_1^{\mathrm{cris}}(X)(R)^i = H_1^{\mathrm{cris}}(X)(R).$$

Because of this, we may compute deformations to PD rings by examining each eigenspace $H_1^{\mathrm{cris}}(X)^i$ separately. This program is carried out below. The reader is invited to perform these computations for herself, possibly after glancing briefly at the exposition given here.

Some notation is necessary to state the result of this calculation. As usual, let $M = \mathbb{D}_*(X) = \oplus M^i$. For $i \in \mathbb{Z}/f\mathbb{Z}$, let $R^i = k[\![\alpha_{jl}^i]\!]_{1 \le j,l \le r}$. For $1 \le j < l \le r$ define certain power series f_{jl}^i in the following way.

If M^i is nice, set

$$f_{jl}^i = \begin{cases} \alpha_{jl}^i - \alpha_{lj}^i & 1 \le j < l \le r-1 \\ \alpha_{lj}^i & 1 \le j \le r-1, l = r \end{cases}.$$

If M^i is not nice, set

$$f_{jl}^i = \begin{cases} \alpha_{12}^i \alpha_{21}^i - \alpha_{11}^i \alpha_{22}^i & j = 1, l = 2 \\ \alpha_{j1}^i \alpha_{l2}^i - \alpha_{j2}^i \alpha_{l1}^i - \alpha_{lj}^i & 1 \le l < 3 \le j \le r \\ \alpha_{lj}^i - \alpha_{j1}^i \alpha_{l2}^i + \alpha_{j2}^i \alpha_{l1}^i - \alpha_{jl}^i & 3 \le l < j \le r \end{cases}.$$

Let $\mathfrak{m}^i \subset R^i$ be the maximal ideal, and let I^i be the ideal generated by the f_{jl}^i.

Lemma 3.2.1

$$\widehat{O}_{\mathcal{A}_{g,d}^{O_K}, (X, \iota, \lambda)} / \mathfrak{m}_{(X, \iota, \lambda)}^p \cong \widehat{\otimes}_{i \in \mathbb{Z}/f\mathbb{Z}} \frac{R^i}{(I^i, (\mathfrak{m}^i)^p)}.$$

Proof   As promised, the proof is an involved computation in crystalline cohomology. Recall that there is a canonical isomorphism $(VM/pM \subset M/pM) \cong \mathrm{Fil}(X)(k) \subset H_1^{\mathrm{cris}}(X)(k)$.

$\boxed{M^i \text{ nice}}$ Suppose M^i is nice. Using 2.1.2, we may choose a normal form for M^i; $M^i = W(k)\{x_1^i, \cdots, x_r^i, y_1^i, \cdots, y_r^i\}$, where

$$\begin{aligned}
Fx_j^i &\in M^{i+1} - pM^{i+1} \\
Fy_j^i &\in pM^{i+1} \\
\langle x_j^i, x_l^i \rangle &= 0 \\
\langle y_j^i, y_l^i \rangle &= 0
\end{aligned}
\qquad
\langle x_j^i, y_l^i \rangle = \begin{cases} 1 & j = l < r \\ p & j = l = r \\ 0 & j \neq l \end{cases}$$

The filtration on M^i is given by $\mathrm{Fil}(M^i/pM^i) = (VM/pM)^i = k\{y_1^i, \cdots, y_r^i\}$. According to identifications in section 2.2, $\mathrm{Fil}(M^{i*}/pM^{i*}) = (M/VM)^{i\vee} = k\{x_1^{i*}, \cdots, x_r^{i*}\}$.

Up to order $p - 1$, the formal moduli space for the filtered vector space $\mathrm{Fil}(M^i/pM^i) \subset M^i/pM^i$ is $R^i = k[\![\alpha_{jl}^i]\!]_{1 \leq j, l \leq r}/(\alpha_{jl}^i)^p$. The universal filtration, of course, is

$$\widetilde{\mathrm{Fil}}(M/pM)^i = \mathrm{span}\langle y_j^i + \sum_{l=1}^r \alpha_{jl}^i x_l^i \rangle.$$

Similarly, the local moduli space for the filtration on the first homology of the dual abelian variety is $k[\![\beta_{jl}]\!]/(\beta_{jl})^p$, and the filtration which lives over it is

$$\widetilde{\mathrm{Fil}}(M^*/pM^*)^i = \mathrm{span}\langle x_j^{i*} + \sum_{l=1}^r \beta_{jl} y_l^{i*} \rangle.$$

In the present setting these two moduli spaces should be somehow linked; to any algebraic deformation of X corresponds a deformation of $X^\vee$, so that $X^\vee$ truly does remain the dual abelian scheme. The important condition is that

$$\langle \widetilde{\mathrm{Fil}}(M/pM)^i, \widetilde{\mathrm{Fil}}(M^*/pM^*)^i \rangle = (0).$$

This imposes certain relations, e.g.,

$$\begin{aligned}
0 &= \langle y_j^i + \sum \alpha_{jl}^i x_l^i, x_{j'}^{i*} + \sum_{l'} \beta_{j'l'} y_{l'}^{i*} \rangle \\
&= \beta_{j'j} + \alpha_{jj'}^i \\
\beta_{j'j} &= -\alpha_{jj'}^i
\end{aligned}$$

So make these identifications systematically and write

$$\widetilde{\mathrm{Fil}}(M^*/pM^*)^i = \mathrm{span}\langle x_j^{i*} - \sum_{l=1}^r \alpha_{lj}^i y_l^{i*} \rangle.$$

The polarization $X \xrightarrow{\lambda} X^\vee$ induces $M \xrightarrow{\lambda_*} M^*$ and $H_1^{\mathrm{cris}}(X) \xrightarrow{\lambda_*} H_1^{\mathrm{cris}}(X^\vee)$. If the deformation is algebraic, it must be a map of filtered crystals; $\lambda_*(\mathrm{Fil}(X)^i) \subseteq \mathrm{Fil}(X^\vee)^i$.

For $1 \leq j \leq r - 1$, $\lambda_*(y_j^i + \sum \alpha_{jl}^i x_l^i) = -x_j^{i*} + \sum_{l=1}^{r-1} \alpha_{jl}^i y_l^{i*} + p\alpha_{jr}^i y_r^{i*}$. If this is to lie in $\widetilde{\mathrm{Fil}}(X)^i$, then

$$-x_j^{i*} + \sum_{l=1}^{r-1} \alpha_{jl}^i y_l^{i*} + p\alpha_{jr}^i y_r^{i*} = -x_j^{i*} + \sum_{l=1}^{r} \alpha_{lj}^i y_l^{i*}.$$

Equate coefficients of y_l^{i*} to find that

$$\boxed{\alpha_{jl}^i = \alpha_{lj}^i} \quad 1 \leq j < l \leq r - 1$$
$$\boxed{\alpha_{rj}^i = 0} \quad 1 \leq j \leq r - 1$$

Similarly, $\lambda_*(y_r^i + \sum \alpha_{rl}^i x_l^i) = \sum_{l=1}^{r-1} \alpha_{rl}^i y_l^{i*}$ mod p, which again forces $\alpha_{rl}^i = 0$ for $1 \leq l \leq r - 1$.

This gives the leading terms of certain local equations for the moduli space at $(X/k, \iota, \lambda)$. We will see shortly that these represent all the equations.

$\boxed{M^i \text{ not nice}}$ Not surprisingly, a similar methodology computes local equations for the non-nice eigenspaces. This time, choose a normal form for $M^i = W(k)\{x_1^i, \cdots, x_r^i, y_1^i, \cdots, y_r^i\}$, where

$$\begin{aligned}
Fx_j^i &\in M^{i+1} - pM^{i+1} \\
Fy_j^i &\in pM^{i+1} \\
\langle x_1^i, x_2^i \rangle &= 1 \\
\langle y_1^i, y_2^i \rangle &= p \\
\langle x_j^i, y_j^i \rangle &= 1 \quad (3 \leq j \leq r)
\end{aligned}$$

and all other elements pair to zero. Again, the universal filtrations are given by

$$\begin{aligned}
\widetilde{\mathrm{Fil}}(X)^i &= \mathrm{span}\langle y_j^i + \sum_{l=1}^{r} \alpha_{jl}^i x_l^i \rangle \\
\widetilde{\mathrm{Fil}}(X^\vee)^i &= \mathrm{span}\langle x_j^{i*} - \sum_{l=1}^{r} \alpha_{lj}^i y_l^{i*} \rangle
\end{aligned}$$

and we must find the conditions ensuring that $\lambda_*(\widetilde{\mathrm{Fil}}(X)^i) \subseteq \widetilde{\mathrm{Fil}}(X^\vee)^i$. We find that

$$\begin{aligned}
\lambda_*(y_1^i + \sum_{l=1}^{r} \alpha_{1l}^i x_l^i) &= py_2^{i*} + \alpha_{11}^i x_2^{i*} - \alpha_{12}^i x_1^{i*} + \sum_{l=3}^{r} \alpha_{1l}^i y_l^{i*} \\
&= \alpha_{11}^i (x_2^{i*} - \sum_{l=1}^{r} \alpha_{l2}^i y_l^{i*}) - \alpha_{12}^i (x_1^{i*} - \sum_{l=1}^{r} \alpha_{l1}^i y_l^{i*}) \\
&= -\alpha_{12}^i x_1^{i*} + \alpha_{11}^i x_2^{i*} + \sum_{l=1}^{r} (\alpha_{12}^i \alpha_{l1}^i - \alpha_{11}^i \alpha_{l2}^i) y_l^{i*}
\end{aligned}$$

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No relation comes from the coefficient of y_1^{i*}, but

$$\boxed{\alpha_{12}^i \alpha_{21}^i - \alpha_{11}^i \alpha_{22}^i = 0} \quad l = 2$$
$$\alpha_{1l}^i = \alpha_{12}^i \alpha_{l1}^i - \alpha_{11}^i \alpha_{l2}^i \quad 3 \le l \le r$$

Already, we see that the defect from nice introduces singularities to the moduli space. Similarly, examining $\lambda_*(y_2^i + \sum_{l=1}^r \alpha_{2l}^i x_l^i)$ yields the additional relation

$$\alpha_{2l}^i = \alpha_{22}^i \alpha_{l1}^i - \alpha_{21}^i \alpha_{l2}^i \quad 3 \le l \le r$$

When $3 \le j \le r$, the natural constraint is

$$\begin{aligned}
\lambda_*(y_j^i + \sum_{l=1}^r \alpha_{jl}^i x_l^i) &= -x_j^{i*} + \alpha_{j1}^i x_2^{i*} - \alpha_{j2}^i x_1^{i*} + \sum_{l=3}^r \alpha_{jl}^i y_l^{i*} \\
&= -(x_j^{i*} + \sum_{l=1}^r \alpha_{lj}^i y_l^{i*}) + \alpha_{j1}^i(x_2^{i*} - \sum_{l=1}^r \alpha_{l2}^i y_l^{i*}) - \alpha_{j2}^i(x_1^{i*} - \sum_{l=1}^r \alpha_{l1}^i y_l^{i*})
\end{aligned}$$

Thus, we get local equations

$$\boxed{\alpha_{lj}^i = \alpha_{j1}^i \alpha_{l2}^i - \alpha_{j2}^i \alpha_{l1}^i} \qquad 1 \le l < 3 \le j \le r$$
$$\boxed{\alpha_{jl}^i = \alpha_{lj}^i - \alpha_{j1}^i \alpha_{l2}^i + \alpha_{j2}^i \alpha_{l1}^i} \quad 3 \le l < j \le r$$

We resume the general discussion. Note that, whether or not M^i is nice, the crystalline cohomology sees $\frac{r(r-1)}{2}$ local equations for each $i \in \mathbb{Z}/f\mathbb{Z}$. A priori, it is possible that there are other equations of sufficiently high leading degree that we cannot detect them using crystalline techniques. However, in view of the lower bound 2.3.2, we know that we have seen the avatars of all equations for the local moduli space. $\heartsuit$

Theorem 3.2.2 Assume $p^f || d$. The moduli space $\mathcal{A}_{g,d}^{O_K}$ is a local complete intersection, and the smooth locus is the nice locus. Ordinary points are dense in $\mathcal{A}_{g,d}^{O_K}$.

Proof The lemma shows that any nice point is smooth. Conversely, let J be the formal local ring of $(\mathcal{A}_{g,d}^{O_K})_{\text{red}}$ at a non-nice point (X, ι, λ). Lemma 3.2.1 gives the initial forms of elements $f_j^i \in k[\![\alpha_{jl}^i]\!]$ presenting J, one of which has the form

$$\alpha_{12}^i \alpha_{21}^i - \alpha_{11}^i \alpha_{22}^i + \text{higher order terms} = 0.$$

It is conceivable that there are additional relations in $\text{rad}(J)$; but if they were linear or quadratic, then the dimension of $O_{\mathcal{A}_{g,d}^{O_K},(X,\iota,\lambda)}$ would drop below the lower bound guaranteed by 2.3.2. Thus, any non-nice point is singular in $(\mathcal{A}_{g,d}^{O_K})_{\text{red}}$, and *a fortiori* in $\mathcal{A}_{g,d}^{O_K}$.

The leading terms computed above tell us a little bit more about the (formal) local structure of $\mathcal{A}_{g,d}^{O_K}$. At any fixed k-point, lemma 3.2.1 provides the initial forms of $f\frac{r(r-1)}{2}$ equations. The tangent cone is the spectrum of a quotient of $k[\![\alpha_{jl}^i]\!]/(I^i)$, the algebra defined by these initial forms. However, since this ring already has the minimum allowed dimension, it must actually be the ring of functions of the tangent cone. In particular, the local ring is a local complete intersection, and it is an integral domain since the tangent cone is one, too. (Still more particularly, the local ring is reduced, and $\mathcal{A}_{g,d}^{O_K}$ is a reduced scheme.)

Finally, given the density of the smooth – and thus the nice – locus, lemma 3.1.1 lets us deduce the density of the ordinary locus.♡

Remark 3.2.3 As mentioned in the introduction, this result gives evidence for a much more general (albeit somewhat less precise) conjecture of Rapoport and Zink. For a broad class of Shimura varieties of PEL type, Rapoport and Zink conjecture [17, p.95] the existence of a flat local model over O_E, the ring of integers of the completion of the reflex field at p. This has been verified in many cases, see especially [5] for a discussion of the Hilbert-Siegel case.

Rapoport and Zink consider local models for moduli spaces of chains of lattices equipped with a skew-symmetric pairing; *grosso modo*, these correspond to Dieudonné modules of p-divisible groups of abelian varieties, and the elementary divisors of one lattice in the next define a parahoric level structure.

Let (X, ι, λ) be a point in $\mathcal{A}_{g,d}^{O_K}$. There is a self-dual lattice inside $\mathbb{D}_*(X[p^\infty])$; the relation between it and the Dieudonné module of X depends on the structure of the polarization. In this way, the local deformation space of (X, ι, λ) may be modelled on one of the local PEL problems of [17].

Now, Görtz proves the conjecture of Rapoport and Zink for a class of examples which essentially subsumes the situation of 3.2.2. Still, it may be worth noting that the present, more precise description of the local rings of $\mathcal{A}_{g,d}^{O_K}$ implies the conjecture. As discussed in the proof of 3.2.2, the special fiber of $\widetilde{\mathcal{A}}_{g,d}^{O_K}$ is reduced. Moreover, by 2.3.3, the closure of the generic fiber $\widetilde{\mathcal{A}}_{g,d}^{O_K}$ is the entire moduli space. Thus, $\widetilde{\mathcal{A}}_{g,d}^{O_K}$ is flat over $W(k)$, as predicted by [17].

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