DISTINGUISHED MODELS OF INTERMEDIATE JACOBIANS

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Abstract. We show that the image of the Abel–Jacobi map admits a model over the field of definition, with the property that the Abel–Jacobi map is equivariant with respect to this model. The cohomology of this abelian variety over the base field is isomorphic as a Galois representation to the deepest part of the coniveau filtration of the cohomology of the projective variety. Moreover, we show that this model over the base field is dominated by the Albanese variety of a product of components of the Hilbert scheme of the projective variety, and thus we answer a question of Mazur. We also recover a result of Deligne on complete intersections of Hodge level one.

Introduction

Let X be a smooth projective variety defined over the complex numbers. Given a nonnegative integer n, denote $\mathrm{CH}^{n+1}(X)$ the Chow group of codimension-$(n+1)$ cycle classes on X, and denote $\mathrm{CH}^{n+1}(X)_{\mathrm{hom}}$ the kernel of the cycle class map $\mathrm{CH}^{n+1}(X) \to H^{2(n+1)}(X, \mathbb{Z}(n+1))$. In the seminal paper [Gri69], Griffiths defined a complex torus, the *intermediate Jacobian*, $J^{2n+1}(X)$ together with an *Abel–Jacobi map*

$$AJ : \mathrm{CH}^{n+1}(X)_{\mathrm{hom}} \to J^{2n+1}(X).$$

While $J^{2n+1}(X)$ and the Abel–Jacobi map are transcendental in nature, the image of the Abel–Jacobi map restricted to $\mathrm{A}^{n+1}(X)$, the sub-group of $\mathrm{CH}^{n+1}(X)$ consisting of algebraically trivial cycle classes, is a complex sub-torus $J^{2n+1}_a(X)$ of $J^{2n+1}(X)$ that is naturally endowed via the Hodge bilinear relations with a polarization, and hence is a complex abelian variety. The first cohomology group of $J^{2n+1}_a(X)$ is naturally identified via the polarization with $\mathrm{N}^n H^{2n+1}(X, \mathbb{Q}(n))$; i.e., the n-th Tate twist of the n-th step in the geometric coniveau filtration (see (1.1)).

If now X is a smooth projective variety defined over a sub-field $K \subseteq \mathbb{C}$, it is natural to ask whether the complex abelian variety $J^{2n+1}_a(X_{\mathbb{C}})$ admits a model over K. In this paper, we prove that $J^{2n+1}_a(X_{\mathbb{C}})$ admits a unique model over K such that

$$AJ : \mathrm{A}^{n+1}(X_{\mathbb{C}}) \to J^{2n+1}_a(X_{\mathbb{C}})$$

is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant, thereby generalizing the well-known cases of the Albanese map $\mathrm{A}^{\dim X}(X_{\mathbb{C}}) \to \mathrm{Alb}_{X_{\mathbb{C}}}$ and of the Picard map $\mathrm{A}^1(X_{\mathbb{C}}) \to \mathrm{Pic}^0_{X_{\mathbb{C}}}$.

Theorem A. *Suppose X is a smooth projective variety over a field $K \subseteq \mathbb{C}$, and let n be a non-negative integer. Then $J^{2n+1}_a(X_{\mathbb{C}})$, the complex abelian variety that is the image of the Abel–Jacobi map $AJ : \mathrm{A}^{n+1}(X_{\mathbb{C}}) \to J^{2n+1}(X_{\mathbb{C}})$, admits a distinguished model J over K such that the Abel–Jacobi map is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. Moreover, there is an algebraic correspondence $\Gamma \in \mathrm{CH}^{\dim(J)+n}(J \times_K X)$ inducing, for every prime number ℓ, a split inclusion of $\mathrm{Gal}(K)$-representations*

$$\Gamma_* : H^1(J_{\overline{K}}, \mathbb{Q}_\ell) \hookrightarrow H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) \tag{0.1}$$

with image $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$.

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The proof is broken into three parts. First we give a proof of Theorem A, up to the statement of the splitting of the inclusion, and where we focus only on the $\mathrm{Aut}(\mathbb{C}/K)$-equivariance of the Abel–Jacobi map on torsion (Theorem 2.1). The proof of Theorem 2.1 relies on showing that $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ is spanned via the action of a correspondence over K by the first cohomology group of a pointed, geometrically integral curve ; this is proved in Proposition 1.1. Next, in §3, we show that if the Abel–Jacobi map is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant on torsion, it is equivariant. This is a consequence of more general results we establish for surjective regular homomorphisms. Finally, the splitting of (0.1) is then proved in Theorem 4.2 as a consequence of Yves André's theory of *motivated cycles* [And96]. Theorem A generalizes [ACMV16a, Thm. A and Thm. B], where the case $n = 1$ was treated. Note however that when $n = 1$ the results of [ACMV16a] are more precise in that the splitting of (0.1) is shown to be induced by an algebraic correspondence over K. Moreover, [ACMV16a, Thm. 4.4] gives a positive characteristic version of Theorem A when $n = 1$.

As a first application of Theorem A, we recover a result of Deligne [Del72] regarding intermediate Jacobians of complete intersections of Hodge level 1 (§5).

Another application is to the following question due to Barry Mazur. Given an effective polarizable weight-1 $\mathbb{Q}$-Hodge structure V, there is a complex abelian variety A (determined up to isogeny) so that $H^1(A, \mathbb{Q})$ gives a weight-1 $\mathbb{Q}$-Hodge structure isomorphic to V. On the other hand, let K be a field, and let ℓ be a prime number (not equal to the characteristic of the field). It is not known (even for $K = \mathbb{Q}$) whether given an effective polarizable weight-1 $\mathrm{Gal}(K)$-representation V_ℓ over $\mathbb{Q}_\ell$, there is an abelian variety A/K such that $H^1(A_{\overline{K}}, \mathbb{Q}_\ell)$ is isomorphic to V_ℓ. A *phantom abelian variety for* V_ℓ is an abelian variety A/K together with an isomorphism of $\mathrm{Gal}(K)$-representations

$$H^1(A_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\cong} V_\ell.$$

Such an abelian variety, if it exists, is determined up to isogeny ; this is called the *phantom isogeny class for* V_ℓ. Mazur asks the following question [Maz14, p.38] : *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$, and let n be a nonnegative integer. If $H^{2n+1}(X_\mathbb{C}, \mathbb{Q})$ has Hodge coniveau n (i.e., $H^{2n+1}(X_\mathbb{C}, \mathbb{C}) = H^{n,n+1}(X) \oplus H^{n+1,n}(X)$), does there exist a phantom abelian variety for $H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$?*

Theorem A answers this affirmatively under the stronger, but according to the generalized Hodge conjecture equivalent, assumption that the Abel–Jacobi map $AJ : \mathrm{A}^{n+1}(X_\mathbb{C}) \to J^{2n+1}(X_\mathbb{C})$ is surjective. This assumption is known to hold in many cases (e.g., uniruled threefolds). Theorem A in fact shows a stronger statement, namely that the *distinguished* model over K of the image of the Abel–Jacobi map $AJ : \mathrm{A}^{n+1}(X_\mathbb{C}) \to J^{2n+1}(X_\mathbb{C})$ provides a phantom abelian variety for the $\mathrm{Gal}(K)$-representation $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$. Moreover, the arguments via motivated cycles of Section 4 give a second proof of the existence of a phantom abelian variety, although not the descent of the image of the Abel–Jacobi map.

Over the complex numbers the image of the Abel–Jacobi map is dominated by Albaneses of resolutions of singularities of products of irreducible components of Hilbert schemes. Since Hilbert schemes are functorial, and in particular defined over K, and since the image of the Abel–Jacobi map descends to K, one might expect the phantom abelian variety to be linked to the Albanese of a Hilbert scheme. Motivated by concrete examples where this holds (e.g., the intermediate Jacobian of a smooth cubic threefold X is the Albanese variety of the Fano variety of lines on X [CG72]), Mazur asks the following question [Maz14, Que. 1] : *Can this phantom abelian variety be constructed as – or at least in terms of – the Albanese variety of some Hilbert scheme geometrically attached to X ?* We provide an affirmative answer for $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$:

Theorem B. *Suppose X is a smooth projective variety over a field $K \subseteq \mathbb{C}$. Then the phantom abelian variety J/K for $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ given in Theorem A is dominated by the Albanese* *variety of (a finite product of resolutions of singularities of some finite number of) components of a Hilbert scheme parameterizing codimension-$(n+1)$ subschemes of X over K.*

The proof of the theorem, given in §6 (Theorem 6.3), is framed in the language of Galois-equivariant regular homomorphisms, as described in [ACMV16a, §4]. As a consequence, some related results are obtained for algebraic representatives of smooth projective varieties over perfect fields of arbitrary characteristic.

For concreteness, we mention the following consequence of Theorems A and B (see §6):

Corollary C. *Suppose X is a smooth projective threefold over a field $K \subseteq \mathbb{C}$ and assume that $X_{\mathbb{C}}$ is uniruled. Then the intermediate Jacobian $J^3(X_{\mathbb{C}})$ descends to an abelian variety J/K, which is a phantom abelian variety for $H^3(X_{\overline{K}}, \mathbb{Q}_\ell(1))$, and is dominated by the Albanese variety of (a product of resolutions of singularities of a finite number of) components of a Hilbert scheme parameterizing dimension-1 subschemes of X over K.*

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Conventions. We use the same conventions as in [ACMV16a]. A *variety* over a field is a geometrically reduced separated scheme of finite type over that field. A *curve* (resp. *surface*) is a variety of pure dimension 1 (resp. 2). Given a variety X, $\mathrm{CH}^i(X)$ denotes the Chow group of codimension i cycles modulo rational equivalence, and $\mathrm{A}^i(X) \subseteq \mathrm{CH}^i(X)$ denotes the subgroup of cycles algebraically equivalent to 0. If X is a smooth projective variety over the complex numbers, then we denote $J^{2n+1}(X)$ the complex torus that is the $(2n+1)$-th intermediate Jacobian of X and we denote $J_a^{2n+1}(X)$ the image of the Abel–Jacobi map $\mathrm{A}^{n+1}(X) \to J^{2n+1}(X)$; a choice of polarization on X naturally endows the complex torus $J_a^{2n+1}(X)$ with the structure of a polarized complex abelian variety. If C/K is a smooth projective geometrically irreducible curve over a field, we will sometimes write $J(C)$ for $\mathrm{Pic}^\circ_{C/K}$. Finally, given an abelian group A, we denote by $A[N]$ the kernel of the multiplication-by-N map; and if A is an abelian group scheme over a field K, we write $A[N]$ for $A(\overline{K})[N]$.

1. A result on cohomology

The main point of this section is to prove Proposition 1.1, strengthening [ACMV16a, Prop. 1.3]. Recall that if X is a smooth projective variety over a field K, then the geometric coniveau filtration $\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is defined by:

$$\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell) := \sum_{\substack{Z \subseteq X \\ \text{closed, codim } \geq \nu}} \ker\left(H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^i(X_{\overline{K}} \backslash Z_{\overline{K}}, \mathbb{Q}_\ell)\right). \tag{1.1}$$

If $K = \mathbb{C}$, the geometric coniveau filtration $\mathrm{N}^\nu H^i(X, \mathbb{Q})$ is defined similarly. We direct the reader to [ACMV16b, §1.2] for a review of some of the properties we use here. Sometimes, we will abuse notation slightly and denote the r-th Tate twist of step ν in the geometric coniveau filtration by $\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell(r)) := (\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell)) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(r)$, and similarly for Betti cohomology.

Proposition 1.1. *Suppose X is a smooth projective variety over a field $K \subseteq \mathbb{C}$, and let n be a nonnegative integer. Then there exist a geometrically integral smooth projective curve C over K, admitting a K-point, and a correspondence $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)_{\mathbb{Q}}$ such that for all primes ℓ, the induced morphism of $\mathrm{Gal}(K)$-representations*

$$\gamma_* : H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \to H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$$

has image $\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$*. Likewise, the morphism of Hodge structures*

$$\gamma_* : H^1(C_{\mathbb{C}}, \mathbb{Q}) \to H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$$

has image $\mathrm{N}^n\, H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$ *; in particular, the image of* $\gamma_* : J(C_{\mathbb{C}}) \to J^{2n+1}(X_{\mathbb{C}})$ *is* $J_a^{2n+1}(X_{\mathbb{C}})$*.*

Remark 1.2. The result [ACMV16a, Prop. 1.3] differs from Proposition 1.1 only in the sense that is it not shown there that C can be taken to admit a K-rational point or to be geometrically integral.

There are three main ingredients in the proof of Proposition 1.1: the Bertini theorems, the Lefschetz type result in Lemma 1.3 below describing cohomology in degree 1, and Proposition A.1 regarding the cohomology of curves. While we expect Proposition A.1 is well-known, for lack of a reference we include a proof in Appendix A.

Lemma 1.3 (Lefschetz)**.** *Suppose* X *is a smooth projective variety over a field* K*. There exist a smooth curve* $C \hookrightarrow X$ *over* K*, which is a (general) linear section for an appropriate projective embedding of* X*, and a correspondence* $\gamma \in \mathrm{CH}^1(C \times_K X)_{\mathbb{Q}}$ *such that for all* $\ell \neq \mathrm{char}(K)$*, the induced morphism of* $\mathrm{Gal}(K)$*-representations*

$$\gamma_* : H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \twoheadrightarrow H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$$

is surjective. Moreover, if X *is geometrically integral (resp. admits a* K*-point), then* C *can be taken to be geometrically integral (resp. to admit a* K*-point).*

Proof. By Bertini [Poo04], let $\iota : C \hookrightarrow X$ be a one-dimensional smooth general linear section of an appropriate projective embedding of X. Note that by the irreducible Bertini theorems [CP16], if X is geometrically integral (resp. admits a K-point), then C can also be taken to be geometrically integral (resp. to admit a K-point).

The hard Lefschetz theorem [Del80, Thm. 4.1.1] states that intersecting with C yields an isomorphism

$$\iota_*\iota^* : H^1(X_{\overline{K}}, \mathbb{Q}_\ell) \hookrightarrow H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \twoheadrightarrow H^{2\dim X-1}(X_{\overline{K}}, \mathbb{Q}_\ell(\dim X - 1)).$$

The Lefschetz Standard Conjecture is known for ℓ-adic cohomology and for Betti cohomology in degree ≤ 1 (see [Kle68, Thm. 2A9(5)]), meaning in our case that the map $(\iota_*\iota^*)^{-1}$ is induced by a correspondence, say $\Lambda \in \mathrm{CH}^1(X \times_K X)_{\mathbb{Q}}$. Therefore, the composition

$$H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \underset{\iota_*}{\overset{(\Gamma_\iota)_*}{\twoheadrightarrow}} H^{2\dim X-1}(X_{\overline{K}}, \mathbb{Q}_\ell(\dim X - 1)) \underset{\cong}{\xrightarrow{\Lambda_*}} H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$$

is surjective and is induced by the correspondence $\gamma := \Lambda \circ \Gamma_\iota$, where Γ_ι denotes the graph of ι . ☐

Proof of Proposition 1.1. Up to working component-wise, we can and do assume that X is irreducible, say of dimension d_X. Since $K \subseteq \mathbb{C}$, we have from the characterization of coniveau (see e.g., [ACMV16a, (1.2)]) that there exist a smooth projective variety Y (possibly disconnected) of pure dimension $d_Y = d_X - n$ over K, and a K-morphism $f : Y \to X$ such that

$$\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) = \mathrm{Im}\left(f_* : H^1(Y_{\overline{K}}, \mathbb{Q}_\ell) \to H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))\right).$$

Using Lemma 1.3 applied to Y, there exist a smooth projective curve C over K (possibly disconnected) and a correspondence $\Gamma \in \mathrm{CH}^1(C \times_K X)_{\mathbb{Q}}$ such that the composition

$$H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \overset{\Gamma_*}{\twoheadrightarrow} H^1(Y_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{f_*} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$$

has image $\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$.

As recalled in Proposition A.1, there is a morphism $\beta : C \to \mathrm{Pic}^\circ_{C/K}$ inducing an isomorphism $\beta^* = (\Gamma^t_\beta)_* : H^1(\mathrm{Pic}^\circ_{C_{\overline{K}}/\overline{K}}, \mathbb{Q}_\ell) \to H^1(C_{\overline{K}}, \mathbb{Q}_\ell)$. Observe that $\mathrm{Pic}^\circ_{C/K}$ is geometrically integral and admits a K-point. Lemma 1.3 yields a smooth geometrically integral curve D/K endowed with a

K-point, and a surjection $H^1(D_{\overline{K}},\mathbb{Q}_\ell) \twoheadrightarrow H^1(\mathrm{Pic}^\circ_{C_{\overline{K}}/\overline{K}},\mathbb{Q}_\ell)$ induced by a correspondence $\widetilde{\Gamma}$ over K. The composition

$$H^1(D_{\overline{K}},\mathbb{Q}_\ell) \underset{\mathrm{Lef}}{\xrightarrow{\widetilde{\Gamma}_*}} H^1(\mathrm{Pic}^\circ_{C_{\overline{K}}/\overline{K}},\mathbb{Q}_\ell) \underset{\cong}{\xrightarrow{(\Gamma^t_\beta)_*}} H^1(C_{\overline{K}},\mathbb{Q}_\ell) \underset{\mathrm{Lef}}{\xrightarrow{\Gamma_*}} H^1(Y_{\overline{K}},\mathbb{Q}_\ell) \underset{\mathrm{Def}}{\xrightarrow{f_*}} \mathrm{N}^n\, H^{2n+1}(X_{\overline{K}},\mathbb{Q}_\ell(n)),$$

induced by the associated composition of correspondences γ, provides the desired surjection

$$\gamma_* : H^1(D_{\overline{K}},\mathbb{Q}_\ell) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{N}^n\, H^{2n+1}(X_{\overline{K}},\mathbb{Q}_\ell(n)) \ .$$

Finally the comparison isomorphisms in cohomology establish (see e.g., [ACMV16a, §1.1]) that the image of the induced morphism of Hodge structures $\gamma_* : H^1(D_{\mathbb{C}},\mathbb{Q}) \to H^{2n+1}(X_{\mathbb{C}},\mathbb{Q}(n))$ is $\mathrm{N}^n\, H^{2n+1}(X_{\mathbb{C}},\mathbb{Q}(n))$; by functoriality of the intermediate Jacobian, this morphism of Hodge structures is induced by a surjection of abelian varieties $\gamma_* : J(D_{\mathbb{C}}) \twoheadrightarrow J^{2n+1}_a(X_{\mathbb{C}})$. ☐

2. Proof of Theorem A : Part I, descent of the image of the Abel–Jacobi map

In this section we establish the following theorem, proving the first part of Theorem A.

Theorem 2.1. *Suppose X is a smooth projective variety over a field $K \subseteq \mathbb{C}$, and let n be a nonnegative integer. Then the image of the Abel–Jacobi map $J^{2n+1}_a(X_{\mathbb{C}})$ admits a distinguished model J over K such that the induced map $AJ : \mathrm{A}^{n+1}(X_{\mathbb{C}})[N] \to J^{2n+1}_a(X_{\mathbb{C}})[N]$ on N-torsion is* $\mathrm{Aut}(\mathbb{C}/K)$*-equivariant for all positive integers N.*

Moreover, there is a correspondence $\Gamma \in \mathrm{CH}^{\dim(J)+n}(J \times_K X)$ such that for each prime number ℓ, we have that Γ induces an inclusion of $\mathrm{Gal}(K)$*-representations*

$$H^1(J_{\overline{K}},\mathbb{Q}_\ell) \xhookrightarrow{\Gamma_*} H^{2n+1}(X_{\overline{K}},\mathbb{Q}_\ell(n)), \tag{2.1}$$

with image $\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}},\mathbb{Q}_\ell(n))$.

We will prove the theorem in several steps contained in the following subsections.

Remark 2.2. By Chow's rigidity theorem (see [Con06, Thm. 3.19]), an abelian variety $A/\mathbb{C}$ descends to at most one model defined over $\overline{K}$. On the other hand, an abelian variety $A/\overline{K}$ may descend to more than one model defined over K. Nevertheless, there is at most one model of A defined over K that induces a given action of $\mathrm{Gal}(K)$ on the $\overline{K}$-points of A ; since torsion points are dense, it suffices to establish this on torsion. Therefore, since $AJ[N] : \mathrm{A}^{n+1}(X_{\mathbb{C}})[N] \to J^{2n+1}_a(X_{\mathbb{C}})[N]$ is surjective for all N, the abelian variety $J^{2n+1}_a(X_{\mathbb{C}})$ admits at most one structure of a scheme over K such that $AJ[N]$ is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant for all positive integers N. This is the sense in which $J^{2n+1}_a(X_{\mathbb{C}})$ admits a distinguished model over K.

2.1. **Chow rigidity and L/K-trace : descending from $\mathbb{C}$ to $\overline{K}$.** The first step in the proof is to use Chow rigidity and $\mathbb{C}/\overline{K}$-trace to descend the image of the Abel–Jacobi map from $\mathbb{C}$ to $\overline{K}$. We follow the treatment in [Con06], and refer the reader to [ACMV16a, §3.3] where we review the theory in the setting we use here.

For the convenience of the reader, we briefly recall a few points. We focus on the case where L/K is an extension of algebraically closed fields of characteristic 0. First, we reiterate that by Chow's rigidity theorem (see [Con06, Thm. 3.19]), an abelian variety B/L descends to at most one model, up to isomorphism, defined over K. Given an abelian variety B defined over L, while B need not descend to K, there is [Con06, Thm. 6.2, Thm. 6.4, Thm. 6.12, p.72, p.76, Thm. 3.19] an abelian variety $\underline{\underline{B}}$ defined over K equipped with an injective homomorphism of abelian varieties

$$\underline{\underline{B}}_L \xhookrightarrow{\tau} B$$

(together called the L/K-trace) with the property that for any abelian variety A/K, base change gives an identification

$$\begin{gathered}\operatorname{Hom}_{\mathsf{Ab}/K}(A, \underline{\underline{B}}) = \operatorname{Hom}_{\mathsf{Ab}/L}(A_L, B) \\ f \mapsto \tau \circ f_L.\end{gathered}$$

It follows that if there is an abelian variety A/K and a surjective homomorphism $A_L \to B$, then τ is surjective and hence an isomorphism; in other words, B descends to K (as $\underline{\underline{B}}$).

Proof of Theorem 2.1, Step 1: $J_a^{2n+1}(X_{\mathbb{C}})$ descends to $\overline{K}$. In the notation of Theorem 2.1, we wish to show that $J_a^{2n+1}(X_{\mathbb{C}})$ descends to an abelian variety over $\overline{K}$. We have shown in Proposition 1.1 that there exist a smooth projective geometrically integral curve C/K, admitting a K-point, and a correspondence $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)_{\mathbb{Q}}$ which induces a surjection $\gamma_* : J(C_{\mathbb{C}}) \twoheadrightarrow J_a^{2n+1}(X_{\mathbb{C}})$. Thus from the theory of the $(\mathbb{C}/\overline{K})$-trace, and the fact that $J(C_{\mathbb{C}}) = J(C_{\overline{K}})_{\mathbb{C}}$ is defined over $\overline{K}$, $J_a^{2n+1}(X_{\mathbb{C}})$ descends to $\overline{K}$ as its $(\mathbb{C}/\overline{K})$-trace $\underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})$, and there is a surjective homomorphism of abelian varieties over $\overline{K}$

$$J(C_{\overline{K}}) \xrightarrow{\underline{\underline{\gamma_*}}} \mathrel{\mkern-14mu}\twoheadrightarrow \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}}).$$

Moreover, the Abel–Jacobi map on torsion $AJ[N] : \mathrm{A}^{n+1}(X_{\mathbb{C}})[N] \to J_a^{2n+1}(X_{\mathbb{C}})[N]$ is $\operatorname{Aut}(\mathbb{C}/\overline{K})$-equivariant for all positive integers N. Indeed, $\operatorname{Aut}(\mathbb{C}/\overline{K})$ acts trivially on $J_a^{2n+1}(X_{\mathbb{C}})[N] = \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})[N]$ and it also acts trivially on $\mathrm{A}^{n+1}(X_{\mathbb{C}})[N]$ by Lecomte's rigidity theorem [Lec86] (see e.g., [ACMV16a, Thm. 3.8(b)]). $\square$

2.2. **Descending from $\overline{K}$ to K.** In the notation of Theorem 2.1, we have found a smooth projective geometrically integral curve C/K, admitting a K-point, and a correspondence $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)_{\mathbb{Q}}$ inducing a surjective homomorphism of abelian varieties over $\overline{K}$

$$0 \longrightarrow P \longrightarrow J(C_{\overline{K}}) \xrightarrow{\underline{\underline{\gamma_*}}} \mathrel{\mkern-14mu}\twoheadrightarrow \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}}) \longrightarrow 0 \tag{2.2}$$

where P is defined to be the kernel. We will show that $\underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})$ descends to K by showing that P descends to K. Since torsion points are Zariski dense in any abelian variety or étale group scheme over an algebraically closed field, we have the following elementary criterion:

Lemma 2.3. *Let A/K be an abelian variety over a perfect field, let Ω/K be an algebraically closed extension field, and let $\overline{A} = A_\Omega$. Suppose that $\overline{B} \subset \overline{A}$ is a closed sub-group scheme. Then $\overline{B}_{\mathrm{red}}$ descends to a sub-group scheme over K if and only if, for each natural number N, we have $\overline{B}[N](\Omega)$ is stable under* $\operatorname{Aut}(\Omega/K)$. $\square$

Proof of Theorem 2.1, Step 2: $\underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})$ descends to K. We wish to show that the abelian variety $\underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})$ over $\overline{K}$, obtained in Step 1 of the proof, descends to an abelian variety over K. In the notation of Step 1, let P be the kernel of $\underline{\underline{\gamma_*}}$, as in (2.2). We use the criterion of Lemma 2.3 to show that P descends to K. To this end, let N be a natural number. We have a commutative diagram of abelian groups:

$$
\begin{tikzcd}
 & & P[N] \arrow[dd, hook] & & \\
P_{\mathbb{C}}[N] \arrow[urr, equal] \arrow[dd, hook] & & & & \\
 & & J(C_{\overline{K}})[N] \arrow[rr, "\simeq"] \arrow[dd, "\underline{\underline{\gamma}}_{*,N}"] & & H^1_{\text{ét}}(C_{\overline{K}}, \boldsymbol{\mu}_N) \arrow[dd, "\gamma_{*,N}"] \\
J(C_{\mathbb{C}})[N] \arrow[urr, equal] \arrow[rrr, "\simeq"'] \arrow[dd, "\gamma_{*,N}"'] & & & H^1_{\text{an}}(C_{\mathbb{C}}, \boldsymbol{\mu}_N) \arrow[ur, equal] \arrow[dd, "\gamma_{*,N}"'] & \\
 & & \underline{\underline{J_a}}^{2n+1}(X_{\mathbb{C}})[N] & & H^{2n+1}_{\text{ét}}(X_{\overline{K}}, \boldsymbol{\mu}_N^{\otimes(n+1)}) \\
J_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[urr, equal] \arrow[r, hook] & J^{2n+1}(X_{\mathbb{C}})[N] \arrow[rr, hook] & & H^{2n+1}_{\text{an}}(X_{\mathbb{C}}, \boldsymbol{\mu}_N^{\otimes(n+1)}) \arrow[ur, equal] &
\end{tikzcd}
\tag{2.3}
$$

The key point is that, by commutativity, the composition of arrows along the back of the diagram

$$
J(C_{\overline{K}})[N] \xrightarrow{\simeq} H^1_{\text{ét}}(C_{\overline{K}}, \boldsymbol{\mu}_N) \xrightarrow{\gamma_{*,N}} H^{2n+1}_{\text{ét}}(X_{\overline{K}}, \boldsymbol{\mu}_N^{\otimes(n+1)})
\tag{2.4}
$$

has the same kernel as the arrow $\underline{\underline{\gamma}}_{*,N}$, namely $P[N]$. Moreover, each arrow of the composition (2.4) is $\mathrm{Gal}(K)$-equivariant. Therefore, $P[N] = \ker \underline{\underline{\gamma}}_{*,N}$ is $\mathrm{Gal}(K)$-stable for each N, and P descends to K. □

2.3. **The Abel–Jacobi map is Galois-equivariant on torsion.** In the notation of Theorem 2.1, we have so far established that $J_a^{2n+1}(X_{\mathbb{C}})$ descends to an abelian variety J over K. We now wish to show that with respect to this given structure as a K-scheme, the Abel–Jacobi map on torsion

$$
AJ : \mathrm{A}^{n+1}(X_{\mathbb{C}})[N] \longrightarrow J_a^{2n+1}(X_{\mathbb{C}})[N] = J[N]
\tag{2.5}
$$

is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. In Step 1, we already showed that AJ is $\mathrm{Aut}(\mathbb{C}/\overline{K})$-equivariant when restricted to torsion. Therefore, in order to conclude, it only remains to prove that the map $\underline{\underline{AJ}} : \mathrm{A}^{n+1}(X_{\overline{K}})[N] \to J[N]$ is $\mathrm{Gal}(\overline{K}/K)$-equivariant.

For future reference, we have the following elementary lemma.

Lemma 2.4. *Let G be a group and let A, B, C be G-modules. Let $\phi : A \to B$ and $\psi : B \to C$ be homomorphisms of abelian groups. We have:*

(a) *If ϕ is surjective and if ϕ and $\psi \circ \phi$ are G-equivariant, then ψ is G-equivariant.*
(b) *If ψ is injective and if ψ and $\psi \circ \phi$ are G-equivariant, then ϕ is G-equivariant.* □

Proof of Theorem 2.1, Step 3: The Abel–Jacobi map is equivariant on torsion. Fix J/K to be the model of $J_a^{2n+1}(X_{\mathbb{C}})$ from Step 2. We wish to show that for any positive integer N, the restriction (2.5) of the Abel–Jacobi map to N-torsion is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. As mentioned above, it only remains to prove that the map $\underline{\underline{AJ}} : \mathrm{A}^{n+1}(X_{\overline{K}})[N] \to \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})[N]$ is $\mathrm{Gal}(\overline{K}/K)$-equivariant.

For this, observe that the Bloch map $\lambda^{n+1} : \mathrm{A}^{n+1}(X_{\overline{K}})[N] \longrightarrow H^{2n+1}_{\text{ét}}(X_{\overline{K}}, \boldsymbol{\mu}_N^{\otimes(n+1)})$ is Galois-equivariant, since it is constructed via natural maps of sheaves, all of which have natural Galois actions. Moreover in characteristic 0, by [Blo79, Prop. 3.7] the Bloch map on torsion factors through the Abel–Jacobi map:

$$
\lambda^{n+1} : \mathrm{A}^{n+1}(X_{\overline{K}})[N] \xrightarrow{\underline{\underline{AJ}}[N]} J_{\overline{K}}[N] \hookrightarrow H^{2n+1}_{\text{ét}}(X_{\overline{K}}, \boldsymbol{\mu}_N^{\otimes(n+1)}).
$$

As described in (2.3), the inclusion $J_{\overline{K}}[N] \hookrightarrow H^{2n+1}_{\text{ét}}(X_{\overline{K}}, \boldsymbol{\mu}_N^{\otimes(n+1)})$ is also Galois-equivariant. By Lemma 2.4(b), we find that $\underline{\underline{AJ}}[N]$ is Galois-equivariant. □

2.4. **The Galois representation.** We now conclude the proof of Theorem 2.1 by constructing the correspondence $\Gamma \in \mathrm{CH}^{\dim(J)+n}(J \times_K X)$ inducing the desired morphism of Galois representations.

Proof of Theorem 2.1, Step 4: The Galois representation. Let J/K be the model of $J_a^{2n+1}(X_{\mathbb{C}})$ from Step 2 (which was shown to be distinguished in Step 3; see Remark 2.2). We will now construct a correspondence $\Gamma \in \mathrm{CH}^{\dim(J)+n}(J \times_K X)$ such that for each prime number ℓ, the correspondence Γ induces an inclusion of $\mathrm{Gal}(K)$-representations

$$H^1(J_{\overline{K}}, \mathbb{Q}_\ell) \xhookrightarrow{\Gamma_*} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)),$$

with image $\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$.

Let C and $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)_{\mathbb{Q}}$ be the smooth, geometrically integral, pointed projective curve and the correspondence provided by Proposition 1.1. As we have seen (in the proof of Theorem 2.1), γ induces a surjective homomorphism of complex abelian varieties $J(C_{\mathbb{C}}) \to J_a^{2n+1}(X_{\mathbb{C}})$ that descends to a homomorphism $f : J(C) \to J$ of abelian varieties defined over K. Consider then the composite morphism

$$H^1(J_{\overline{K}}, \mathbb{Q}_\ell) \xhookrightarrow{f^*} H^1(J(C)_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow[\simeq]{\mathrm{alb}^*} H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\gamma_*} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)), \tag{2.6}$$

where $\mathrm{alb} : C \to J(C)$ denotes the Albanese morphism induced by the K-point of C. This morphism is clearly injective and induced by a correspondence on $J \times_K X$, and we claim that its image is $\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$. Indeed, the complexification of (2.6) together with the comparison isomorphisms yields a diagram

$$H^1(J_a^{2n+1}(X_{\mathbb{C}}), \mathbb{Q}) \xhookrightarrow{(f_{\mathbb{C}})^*} H^1(J(C)_{\mathbb{C}}, \mathbb{Q}) \xrightarrow[\simeq]{(\mathrm{alb}_{\mathbb{C}})^*} H^1(C_{\mathbb{C}}, \mathbb{Q}) \xrightarrow{(\gamma_{\mathbb{C}})_*} H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n)), \tag{2.7}$$

where $(\mathrm{alb}_{\mathbb{C}})^* \circ (f_{\mathbb{C}})^*$ is easily seen to be the dual (via the natural choice of polarizations) of $(\gamma_{\mathbb{C}})_*$. Since the Hodge structure $H^1(C_{\mathbb{C}}, \mathbb{Q})$ is polarized by the cup-product, we conclude by [ACMV16a, Lemma 2.3] that the image of (2.7) is equal to the image of $(\gamma_{\mathbb{C}})_*$, that is, to $\mathrm{N}^n\, H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$. Invoking the comparison isomorphism settles the claim.

This completes the proof of Theorem 2.1. $\square$

2.5. **A functoriality statement.** Recall that if X and Y are smooth projective varieties over a field $K \subseteq \mathbb{C}$, and $Z \in \mathrm{CH}^{m-n+\dim X}(X \times_K Y)$ is a correspondence, then Z induces functorially a homomorphism of complex abelian varieties

$$\psi_{Z_{\mathbb{C}}} : J_a^{2n+1}(X_{\mathbb{C}}) \to J_a^{2m+1}(Y_{\mathbb{C}})$$

that is compatible with the Abel–Jacobi map.

Proposition 2.5. *Denote J and J' the distinguished models of $J_a^{2n+1}(X_{\mathbb{C}})$ and $J_a^{2m+1}(Y_{\mathbb{C}})$ over K. Then the homomorphism $\psi_{Z_{\mathbb{C}}}$ descends to a K-homomorphism of abelian varieties $\psi_Z : J \to J'$.*

Proof. For each N, the Abel–Jacobi map $AJ : \mathrm{A}^{n+1}(X_{\mathbb{C}}) \to J_a^{2n+1}(X_{\mathbb{C}})$ is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant (Theorem 2.1) and surjective on N-torsion. Applying Lemma 2.4(a) to the commutative square

$$\begin{tikzcd}
\mathrm{A}^{n+1}(X_{\mathbb{C}})[N] \arrow[r, two heads, "AJ[N]"] \arrow[d, "(Z_{\mathbb{C}})_*[N]"'] & J_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, "\psi_{Z_{\mathbb{C}}}[N]"] \\
\mathrm{A}^{m+1}(Y_{\mathbb{C}})[N] \arrow[r, two heads, "AJ[N]"] & J_a^{2m+1}(Y_{\mathbb{C}})[N]
\end{tikzcd}$$

shows that $\psi_{Z_{\mathbb{C}}}[N]$ is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. Since torsion is dense, this implies that $\psi_{Z_{\mathbb{C}}}$ is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant on $\mathbb{C}$-points. From the theory of the $\mathbb{C}/\overline{K}$-trace, $\psi_{Z_{\mathbb{C}}}$ descends to a morphism $\underline{\underline{\psi}}_{Z_{\mathbb{C}}} : \underline{\underline{J}}^{2n+1}_a(X_{\mathbb{C}}) \to \underline{\underline{J}}^{2m+1}_a(Y_{\mathbb{C}})$ over $\overline{K}$. Then the $\mathrm{Aut}(\mathbb{C}/K)$-equivariance of $\psi_{Z_{\mathbb{C}}}$ on $\mathbb{C}$-points implies $\underline{\underline{\psi}}_{Z_{\mathbb{C}}}$ is $\mathrm{Gal}(\overline{K}/K)$-equivariant on $\overline{K}$-points, and so descends from $\overline{K}$ to K. □

3. Proof of Theorem A : Part II, regular homomorphisms and torsion points

Given a smooth projective complex variety X, a fundamental result of Griffiths [Gri69] is that the Abel–Jacobi map $AJ : \mathrm{A}^{n+1}(X) \longrightarrow J^{2n+1}_a(X)$ is a *regular homomorphism*. This means that for every pair (T, Z) with T a pointed smooth integral complex variety, and $Z \in \mathrm{CH}^i(T \times X)$, the composition

$$T(\mathbb{C}) \xrightarrow{\ w_Z\ } \mathrm{A}^i(X) \xrightarrow{\ \phi\ } J^{2n+1}_a(X)$$

is induced by a morphism of complex varieties $\psi_Z : T \to J^{2n+1}_a(X)$, where, if $t_0 \in T(\mathbb{C})$ is the base point of T, $w_Z : T(\mathbb{C}) \to \mathrm{A}^i(X)$ is given by $t \mapsto Z_t - Z_{t_0}$; here Z_t is the refined Gysin fiber. Likewise, one defines regular homomorphisms for smooth projective varieties defined over an arbitrary algebraically closed field. We direct the reader to [ACMV16a, §3] for a review of the material we use here on regular homomorphisms and *algebraic representatives*, and to [ACMV16a, §4] for the notion of a *Galois-equivariant regular homomorphism*. In this section we provide some results regarding equivariance of regular homomorphisms ; the main results are Propositions 3.4 and 3.7.

3.1. Preliminaries.

We will utilize the following facts :

Proposition 3.1 ([ACMV16b, Thm. 2])**.** *Let X/K be a scheme of finite type over a perfect field K. If $\alpha \in \mathrm{CH}^i(X_{\overline{K}})$ is algebraically trivial, then there exist an abelian variety A/K, a cycle $Z \in \mathrm{CH}^i(A \times_K X)$, and a $\overline{K}$-point $t \in A(\overline{K})$ such that $\alpha = Z_t - Z_0$.* □

Proof. We have shown in [ACMV16b, Thm. 2] that there exist an abelian variety A'/K, a cycle Z' on $A' \times_K X$, and a pair of $\overline{K}$-points $t_1, t_0 \in A'(\overline{K})$ such that $\alpha = Z'_{t_1} - Z'_{t_0}$. Let $p_{13}, p_{23} : A' \times_K A' \times_K X \to A' \times_K X$ be the obvious projections. Let Z be defined as the cycle $Z := p_{13}^* Z' - p_{23}^* Z'$ on $A' \times_K A' \times_K X$. For points $t_1, t_0 \in \underline{A}'(\overline{K})$, we have $Z_{(t_1,t_0)} = Z'_{t_1} - Z'_{t_0}$. Thus setting $A = A' \times_K A'$, we are done. □

Lemma 3.2. *Let X be a scheme of finite type over an algebraically closed field $\overline{K}$, let $B/\overline{K}$ be an abelian variety, and let $Z \in \mathrm{CH}^i(B \times_{\overline{K}} X)$. Then the map $w_Z : B(\overline{K}) \to \mathrm{A}^i(X)$ sends torsion points of $B(\overline{K})$ to torsion cycle classes in $\mathrm{A}^i(X_{\overline{K}})$.*

Proof. Since w_Z factors as $B(\overline{K}) \longrightarrow \mathrm{A}_0(B) \xrightarrow{Z_*} \mathrm{A}^i(X)$, where the first arrow is the map $t \mapsto [t] - [0]$ and the second arrow is the group homomorphism induced by the action of the correspondence Z, it suffices to observe that the map $B(\overline{K}) \to \mathrm{A}_0(B)$ sends torsion points of B to torsion cycles in $\mathrm{A}_0(B)$. This is [Bea83, Prop. 11, Lem. p.259] and [Blo76, Thm. (0.1)]. □

3.2. Algebraically closed base change and equivariance of regular homomorphisms.

In this section we will utilize traces for algebraically closed field extensions in arbitrary characteristic. The main results of this paper focus on the characteristic 0 case, which we reviewed in §2.1. The properties of the trace that we utilize here in positive characteristic are reviewed in [ACMV16a, §3.3.1] ; the main difference is that we must potentially keep track of some purely inseparable isogenies.

Lemma 3.3. *Let Ω/k be an extension of algebraically closed fields, and let X be a smooth projective variety over k. Let A be an abelian variety over Ω and let $\phi : \mathrm{A}^i(X_\Omega) \to A(\Omega)$ be a surjective regular homomorphism. Setting $\tau : \underline{\underline{A}}_\Omega \to A$ to be the Ω/k-trace of A, we have that τ is a purely inseparable isogeny, which is an isomorphism in characteristic* 0. *Moreover, there is a regular homomorphism $(\underline{\underline{\phi}})_\Omega : \mathrm{A}^i(X_\Omega) \to \underline{\underline{A}}_\Omega(\Omega)$ making the following diagram commute*

$$\begin{tikzcd} \mathrm{A}^i(X_\Omega) \arrow[r, "(\underline{\underline{\phi}})_\Omega"] \arrow[d, equal] & \underline{\underline{A}}_\Omega(\Omega) \arrow[d, "\tau(\Omega)", "\simeq"'] \\ \mathrm{A}^i(X_\Omega) \arrow[r, "\phi"] & A(\Omega). \end{tikzcd} \tag{3.1}$$

Proof. Let us start by recalling some of the set-up from [ACMV16a, Thm. 3.7]. First, consider the regular homomorphism $\underline{\underline{\phi}} : \mathrm{A}^i(X) \to \underline{\underline{A}}(k)$ constructed in Step 2 of the proof [ACMV16a, Thm. 3.7]. It fits into a commutative diagram [ACMV16a, (3.9)]

$$\begin{tikzcd} \mathrm{A}^i(X) \arrow[r, "\underline{\underline{\phi}}"] \arrow[dd, "\text{base change}"'] & \underline{\underline{A}}(k) \arrow[d, "\text{base change}"] \\ & \underline{\underline{A}}_\Omega(\Omega) \arrow[d, "\tau(\Omega)"] \\ \mathrm{A}^i(X_\Omega) \arrow[r, "\phi"] & A(\Omega). \end{tikzcd} \tag{3.2}$$

Since we are assuming that $\phi : \mathrm{A}^i(X_\Omega) \to A(\Omega)$ is surjective, Step 3 of the proof of [ACMV16a, Thm. 3.7] yields that $\underline{\underline{\phi}} : \mathrm{A}^i(X) \to \underline{\underline{A}}(k)$ is surjective, and that $\tau : \underline{\underline{A}}_\Omega \to A$ is a purely inseparable isogeny, which is an isomorphism in characteristic 0. In particular, $\tau(\Omega) : \underline{\underline{A}}_\Omega(\Omega) \to A(\Omega)$ is an isomorphism.

Now consider the regular homomorphism $\underline{\underline{\phi}}_\Omega : \mathrm{A}^i(X_\Omega) \to \underline{\underline{A}}_\Omega(\Omega)$ constructed in Step 1 of the proof of [ACMV16a, Thm. 3.7], which by *loc. cit.* is surjective. We can therefore fill in diagram (3.2) to obtain:

$$\begin{tikzcd} \mathrm{A}^i(X) \arrow[r, "\underline{\underline{\phi}}"] \arrow[d, "\text{base change}"'] & \underline{\underline{A}}(k) \arrow[d, "\text{base change}"] \\ \mathrm{A}^i(X_\Omega) \arrow[r, "(\underline{\underline{\phi}})_\Omega"] \arrow[d, equal] & \underline{\underline{A}}_\Omega(\Omega) \arrow[d, "\tau(\Omega)", "\simeq"'] \\ \mathrm{A}^i(X_\Omega) \arrow[r, "\phi"] & A(\Omega). \end{tikzcd} \tag{3.3}$$

We claim that (3.3) is commutative. To start, the commutativity of the top square is established in Step 1 of the proof of [ACMV16a, Thm. 3.7], and we have already confirmed the commutativity of the outer rectangle, above. For the bottom square we argue as follows.

By rigidity for torsion cycles on X ([Jan15, Lec86]; see also [ACMV16a, Thm. 3.8(b)]) and for torsion points on $\underline{\underline{A}}$, the vertical arrows in diagram (3.3) are isomorphisms on torsion. A little more naively (i.e., without using [Jan15]), one can simply fix a prime number ℓ not equal to $\mathrm{char}\, k$, and consider torsion to be ℓ-power torsion, and the rest of the argument goes through without change. The top square and outer rectangle are commutative, and thus (3.3) is commutative on torsion. Now let $\alpha \in \mathrm{A}^i(X_\Omega)$. By Weil [Wei54, Lem. 9] (e.g., Proposition 3.1) there exist an abelian variety B/Ω, a cycle class $Z \in \mathrm{CH}^i(B \times_\Omega X_\Omega)$, and an Ω-point $t \in B(\Omega)$ such that $\alpha = Z_t - Z_0$. Then consider the following diagram (not *a priori* commutative) :

$$
\begin{tikzcd}
B(\Omega) \arrow[r, "w_Z"] \arrow[d, equal] & \mathrm{A}^i(X_\Omega) \arrow[r, "(\underline{\underline{\phi}})_\Omega"] \arrow[d, equal] & \underline{\underline{A}}_\Omega(\Omega) \arrow[d, "\tau(\Omega)", "\simeq"'] \\
B(\Omega) \arrow[r, "w_Z"] & \mathrm{A}^i(X_\Omega) \arrow[r, "\phi"] & A(\Omega).
\end{tikzcd}
\tag{3.4}
$$

The left-hand square is obviously commutative. We have shown that the right-hand square is commutative on torsion. The horizontal arrows on the left send torsion points to torsion cycle classes (Lemma 3.2). Therefore the whole diagram (3.4) is commutative on torsion. Since torsion points are Zariski dense in abelian varieties, the diagram is commutative if we replace $\mathrm{A}^i(X_\Omega)$ with $\mathrm{Im}(w_Z)$. Since $\alpha \in \mathrm{Im}(w_Z)$, we see that $(\tau(\Omega) \circ (\underline{\underline{\phi}})_\Omega)(\alpha) = \phi(\alpha)$. Thus, since α was arbitrary, the lemma is proved. □

Proposition 3.4. *Let Ω/k be an extension of algebraically closed fields of characteristic 0, and let X be a smooth projective variety over k. Let A be an abelian variety over Ω and let $\phi : \mathrm{A}^i(X_\Omega) \to A(\Omega)$ be a surjective regular homomorphism. Then A admits a model over k, the Ω/k-trace of A, such that ϕ is $\mathrm{Aut}(\Omega/k)$-equivariant.*

Proof. This follows directly from Lemma 3.3. Indeed, by the construction of $(\underline{\underline{\phi}})_\Omega$ in Step 1 of [ACMV16a, Thm. 3.7], $(\underline{\underline{\phi}})_\Omega$ is $\mathrm{Aut}(\Omega/k)$-equivariant. Then, since $\tau : \underline{\underline{A}}_\Omega \to A$ is an isomorphism, we are done. □

Remark 3.5. More generally, if $\mathrm{char}\, k \neq 0$, then in the notation of Proposition 3.4, the abelian variety A admits a purely inseparable isogeny to an abelian variety over Ω that descends to k, namely the Ω/k-trace. Moreover, under this purely inseparable isogeny, the Ω-points of both abelian varieties are identified, and under the induced action of $\mathrm{Aut}(\Omega/k)$ on $A(\Omega)$, we have that ϕ is $\mathrm{Aut}(\Omega/k)$-equivariant.

3.3. **Galois-equivariant regular homomorphisms and torsion points.** The main point of this subsection is to prove Proposition 3.7. This allows us to utilize results of [ACMV16a] on regular homomorphisms in the setting of torsion points. We start with the following lemma.

Lemma 3.6. *Let A be an abelian variety over a perfect field K and let $\phi : \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ be a regular homomorphism. Assume that there is a prime $\ell \neq \mathrm{char}(K)$ such that for all positive integers n we have that the map $\phi[\ell^n] : \mathrm{A}^i(X_{\overline{K}})[\ell^n] \to A[\ell^n]$ is $\mathrm{Gal}(K)$-equivariant. Let B/K be an abelian variety and let $Z \in \mathrm{CH}^i(B \times_K X)$ be a cycle class. Then the induced morphism $\psi_{Z_{\overline{K}}} : B_{\overline{K}} \to A_{\overline{K}}$ is defined over K.*

Proof. Since (geometric) ℓ-primary torsion points are Zariski dense in the graph of $\psi_{Z_{\overline{K}}}$ inside $B \times_K A$, it suffices to show that the induced morphism $B(\overline{K}) \to A(\overline{K})$ is Galois-equivariant on ℓ-primary torsion. Since the map $w_Z : B(\overline{K}) \to \mathrm{A}^i(X_{\overline{K}})$ is Galois-equivariant and since $\phi : \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ is Galois-equivariant on ℓ^n-torsion for all positive integers n, it is even enough to show that the map $w_Z : B(\overline{K}) \to \mathrm{A}^i(X_{\overline{K}})$ sends torsion points of $B(\overline{K})$ to torsion cycles in $\mathrm{A}^i(X_{\overline{K}})$. This is Lemma 3.2. □

We can now prove :

Proposition 3.7. *Let A be an abelian variety over a perfect field K and let $\phi : \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ be a regular homomorphism. Assume that there is a prime $\ell \neq \mathrm{char}(K)$ such that for all positive integers n the map $\phi[\ell^n] : \mathrm{A}^i(X_{\overline{K}})[\ell^n] \to A[\ell^n]$ is $\mathrm{Gal}(K)$-equivariant. Then ϕ is $\mathrm{Gal}(K)$-equivariant.*

Proof. Let $\alpha \in \mathrm{A}^i(X_{\overline{K}})$, and let $\sigma \in \mathrm{Gal}(K)$. From Proposition 3.1, we have an abelian variety B/K, a cycle $Z \in \mathrm{CH}^i(B \times_K X)$, and a $\overline{K}$-point $t \in B(\overline{K})$ such that $\alpha = Z_t - Z_0$. Now consider the following diagram (not *a priori* commutative):

$$\begin{array}{ccccc} B(\overline{K}) & \xrightarrow{w_{Z_{\overline{K}}}} & \mathrm{A}^i(X_{\overline{K}}) & \xrightarrow{\phi} & A(\overline{K}) \\ \sigma_B^* \big\downarrow & & \sigma_X^* \big\downarrow & & \sigma_A^* \big\downarrow \\ B(\overline{K}) & \xrightarrow{w_{Z_{\overline{K}}}} & \mathrm{A}^i(X_{\overline{K}}) & \xrightarrow{\phi} & A(\overline{K}). \end{array}$$

Since Z is defined over K, and the base point 0 is defined over K, the left-hand square is commutative (e.g., [ACMV16a, Rem. 4.3]). It follows from Lemma 3.6 that the outer rectangle is also commutative. Therefore, from Lemma 2.4(a), the right-hand square in the diagram is commutative on the image of $w_{Z_{\overline{K}}}$. In particular, $\phi(\sigma_X^*\alpha) = \sigma_B^*\phi(\alpha)$. ☐

4. Proof of Theorem A: Part III, the coniveau filtration is split

We now complete the proof of Theorem A by showing that the coniveau filtration is split (Corollary 4.4). For this purpose, we use Yves André's theory of motivated cycles [And96]. Along the way, we show in Theorem 4.2 that the existence of a phantom isogeny class for $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ for all primes ℓ follows directly from André's theory. Note that we already proved this in Theorem 2.1 in a more precise form, namely by showing that there exists a *distinguished* phantom abelian variety within the isogeny class.

For clarity, we briefly review the setup of André's theory of motivated cycles, and fix some notation. Given a smooth projective variety X over a field K and a prime $\ell \neq \mathrm{char}(K)$, let us denote $\mathrm{B}^j(X)_\mathbb{Q}$ the image of the cycle class map $\mathrm{CH}^j(X)_\mathbb{Q} \to H^{2j}(X_{\overline{K}}, \mathbb{Q}_\ell(j))$. A *motivated cycle* on X with rational coefficients is an element of the graded algebra $\bigoplus_r H^{2r}(X_{\overline{K}}, \mathbb{Q}_\ell(r))$ of the form $\mathrm{pr}_*(\alpha \cup *\beta)$, where α and β are elements of $\mathrm{B}^*(X \times_K Y)_\mathbb{Q}$ with Y an arbitrary smooth projective variety over K, $\mathrm{pr} : X \times_K Y \to X$ is the natural projection, and $*$ is the Lefschetz involution on $\bigoplus_r H^{2r}((X \times_K Y)_{\overline{K}}, \mathbb{Q}_\ell(r))$ relative to any polarization on $X \times_K Y$. The set of motivated cycles on X, denoted $\mathrm{B}^\bullet_{\mathrm{mot}}(X)_\mathbb{Q}$, forms a graded $\mathbb{Q}$-sub-algebra of $\bigoplus_r H^{2r}(X_{\overline{K}}, \mathbb{Q}_\ell(r))$, with $\mathrm{B}^r_{\mathrm{mot}}(X)_\mathbb{Q} \subseteq H^{2r}(X_{\overline{K}}, \mathbb{Q}_\ell(r))$; *cf.* [And96, Prop. 2.1]. Taking $Y = \mathrm{Spec}\, K$ above, we have an inclusion $\mathrm{B}^r(X)_\mathbb{Q} \subseteq \mathrm{B}^r_{\mathrm{mot}}(X)_\mathbb{Q}$. Moreover there is a notion of motivated correspondences between smooth projective varieties, and there is a composition law with the expected properties.

Proposition 4.1. *Let Y and X be smooth projective varieties over a field $K \subseteq \mathbb{C}$. Consider a motivated cycle $\gamma \in \mathrm{B}^{d_Y+r}_{\mathrm{mot}}(Y \times_K X)_\mathbb{Q}$ and its action*

$$\gamma_* : H^j(Y_{\overline{K}}, \mathbb{Q}_\ell) \longrightarrow H^{j+2r}(X_{\overline{K}}, \mathbb{Q}_\ell(r)) .$$

Then $\mathrm{Im}(\gamma_*)$ (resp. $\ker(\gamma_*)$) is a direct summand of the $\mathrm{Gal}(K)$-representation $H^{j+2r}(X_{\overline{K}}, \mathbb{Q}_\ell(r))$ (resp. $H^j(Y_{\overline{K}}, \mathbb{Q}_\ell)$).

Proof. We are going to show that if $\gamma \in \mathrm{B}^{d_Y+r}_{\mathrm{mot}}(Y \times_K X)_\mathbb{Q}$ is a motivated correspondence, then there exists an idempotent motivated correspondence $p \in \mathrm{B}^{d_Y}_{\mathrm{mot}}(Y \times_K Y)_\mathbb{Q}$ such that $p_* H^j(Y_{\overline{K}}, \mathbb{Q}_\ell) = \ker(\gamma_*)$. Assuming the existence of such a p, this would establish that $\ker(\gamma_*)$ is a direct summand of $H^j(Y_{\overline{K}}, \mathbb{Q}_\ell)$ as a $\mathbb{Q}_\ell$-vector space. But then by [And96, Scolie 2.5], motivated cycles on a smooth projective variety Y over K are exactly the $\mathrm{Gal}(K)$-invariant motivated cycles on $Y_{\overline{K}}$; therefore $\ker(\gamma_*)$ is indeed a direct summand of $H^j(Y_{\overline{K}}, \mathbb{Q}_\ell)$ as a $\mathrm{Gal}(K)$-representation, completing the proof. The statement about the image of γ_* follows by duality.

The existence of p follows formally from [And96, Thm. 0.4]: the $\otimes$-category of pure motives $\mathscr{M}$ over a field K of characteristic zero obtained by using motivated correspondences rather than algebraic correspondences is a graded, abelian semi-simple, polarized, and Tannakian category over $\mathbb{Q}$. Indeed, using the notations from [And96, §4] and viewing γ as a morphism from the motive $\mathfrak{h}(Y)$ to the motive $\mathfrak{h}(X)(r)$, we see by semi-simplicity that there exists an idempotent motivated correspondence $p \in \mathrm{B}^{d_Y}_{\mathrm{mot}}(Y \times_K Y)_{\mathbb{Q}}$ such that $\ker(\gamma) = p\mathfrak{h}(Y)$. Now the Tannakian category $\mathscr{M}$ is neutralized by the fiber functor to the category of $\mathbb{Q}_\ell$-vector spaces given by the ℓ-adic realization functor. Since by definition a fiber functor is exact, $p_*H^j(Y_{\overline{K}}, \mathbb{Q}_\ell) = \ker(\gamma_*)$ as $\mathbb{Q}_\ell$-vector spaces. $\square$

Theorem 4.2. *Suppose X is a smooth projective variety over a field $K \subseteq \mathbb{C}$, and let n be a nonnegative integer. The $\mathrm{Gal}(K)$-representation $\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ admits a phantom; more precisely there exist an abelian variety J' over K and a correspondence $\Gamma' \in \mathrm{CH}^{\dim J'+n}(J' \times_K X)$ such that the morphism of $\mathrm{Gal}(K)$-representations*

$$\Gamma'_* : H^1(J'_{\overline{K}}, \mathbb{Q}_\ell) \hookrightarrow H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) \tag{4.1}$$

is split injective with image $\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$.

Proof. Let C and $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)_{\mathbb{Q}}$ be the pointed curve and the correspondence provided by Proposition 1.1. By Proposition 4.1 and its proof, there is an idempotent motivated correspondence $q \in \mathrm{B}^1_{\mathrm{mot}}(C \times_K C)_{\mathbb{Q}}$ such that $q_*H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\gamma_*} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ is a monomorphism of $\mathrm{Gal}(K)$-representations with image $\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$, which is itself a direct summand of $H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$.

Now we claim that for smooth projective varieties defined over a field of characteristic zero, we have $\mathrm{B}^1_{\mathrm{mot}}(-)_{\mathbb{Q}} = \mathrm{B}^1(-)_{\mathbb{Q}}$. Over an algebraically closed field of characteristic zero this is a consequence of the Lefschetz $(1,1)$-theorem. Over a field K of characteristic zero, the claim follows from the following two facts: (1) if Y is a smooth projective variety over K, then $\mathrm{B}^r(Y)_{\mathbb{Q}}$ consists of the $\mathrm{Gal}(K)$-invariant classes in $\mathrm{B}^r(Y_{\overline{K}})_{\mathbb{Q}}$ by a standard norm argument, and similarly (2) $\mathrm{B}^r_{\mathrm{mot}}(Y)_{\mathbb{Q}}$ consists of the $\mathrm{Gal}(K)$-invariant classes in $\mathrm{B}^r_{\mathrm{mot}}(Y_{\overline{K}})_{\mathbb{Q}}$ by [And96, Scolie 2.5].

Therefore the motivated idempotent q is in fact an idempotent correspondence in $\mathrm{B}^1(C \times_K C)_{\mathbb{Q}}$, and thus defines, up to isogeny, an idempotent endomorphism of $\mathrm{Pic}^\circ(C)$. Its image J', which is only defined up to isogeny, is the sought-after abelian variety such that $q_*H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \cong H^1(J'_{\overline{K}}, \mathbb{Q}_\ell)$. Composing the transpose of the graph of the morphism $C \hookrightarrow \mathrm{Pic}^\circ(C) \twoheadrightarrow J'$ with the algebraic correspondence γ yields the desired correspondence $\Gamma' \in \mathrm{CH}^{\dim J'+n}(J' \times_K X)$. $\square$

Remark 4.3. The main difference with [ACMV16a, Thm. 2.1] is that we do not know if the splitting in Theorem 4.2 is induced by an algebraic correspondence over K. In that respect [ACMV16a, Thm. 2.1] is more precise.

A nice consequence of Proposition 4.1 is the following:

Corollary 4.4. *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. The geometric coniveau filtration on the $\mathrm{Gal}(K)$-representation $H^n(X_{\overline{K}}, \mathbb{Q}_\ell)$ is split.*

Proof. Let r be a nonnegative integer. Using the coniveau hypothesis, resolution of singularities, mixed Hodge theory, and comparison isomorphisms, there exist a smooth projective variety Y of dimension $\dim X - r$ over K and a morphism $f : Y \to X$ such that the induced morphism of $\mathrm{Gal}(K)$-representations

$$f_* : H^{n-2r}(Y_{\overline{K}}, \mathbb{Q}_\ell(-r)) \to H^n(X_{\overline{K}}, \mathbb{Q}_\ell)$$

has image $\mathrm{N}^r\, H^n(X_{\overline{K}}, \mathbb{Q}_\ell)$; see e.g. [ACMV16a, (1.2)]. The splitting of the coniveau filtration follows from Proposition 4.1 and the Krull–Schmidt theorem. $\square$

Proof of Theorem A. Everything except for the splitting of the inclusion (0.1) in Theorem A is shown by combining Theorem 2.1 with Propositions 3.4 and 3.7. The splitting follows from Corollary 4.4. $\square$

5. Deligne's theorem on complete intersections of Hodge level 1

We recapture Deligne's result [Del72] on intermediate Jacobians of complete intersections of Hodge level 1 (Deligne's primary motivation was to establish the Weil conjectures for those varieties; of course Deligne established the Weil conjectures in full generality a few years later):

Theorem 5.1 (Deligne [Del72])**.** *Let X be a smooth complete intersection of odd dimension $2n+1$ over a field $K \subseteq \mathbb{C}$. Assume that X has Hodge level 1, that is, assume that $h^{p,q}(X_{\mathbb{C}}) = 0$ for all $|p-q| > 1$. Then the intermediate Jacobian $J^{2n+1}(X_{\mathbb{C}})$ is a complex abelian variety that is defined over K.*

Proof. First note that the assumption that X has Hodge level 1 implies that the cup product on $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Z})$ endows the complex torus $J^{2n+1}(X_{\mathbb{C}})$ with a Riemannian form so that $J^{2n+1}(X_{\mathbb{C}})$ is naturally a principally polarized complex abelian variety. Deligne's proof that this complex abelian variety is defined over K uses the irreducibility of the monodromy action of the fundamental group of the universal deformation of X on $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q})$ and on $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Z}/\ell)$ for all primes ℓ. Here, we give an alternate proof based on our Theorem A.

Denote $V_m(a_1, \dots, a_k)$ a smooth complete intersection of dimension n of multi-degree $(a_1, \dots, a_k)$ inside $\mathbb{P}^{m+k}$. A complete intersection X of Hodge level 1 of odd dimension is of one of the following types: $V_{2n+1}(2), V_{2n+1}(2,2), V_{2n+1}(2,2,2), V_3(3), V_3(2,3), V_5(3), V_3(4)$; see for instance [Rap72, Table 1]. In the cases where X is one of the above and X has dimension 3, then X is Fano and as such is rationally connected, and therefore $\mathrm{CH}_0(X_{\mathbb{C}}) = \mathbb{Z}$. In all of the other listed cases, it is known [Otw99, Cor. 1] that $\mathrm{CH}_0(X_{\mathbb{C}})_{\mathbb{Q}}, \dots, \mathrm{CH}_{n-1}(X_{\mathbb{C}})_{\mathbb{Q}}$ are spanned by linear sections. A decomposition of the diagonal argument [BS83] then shows that if X is a complete intersection of Hodge level 1, then the Abel–Jacobi map $\mathrm{A}^n(X_{\mathbb{C}}) \to J^{2n+1}(X_{\mathbb{C}})$ is surjective, i.e., that $J^{2n+1}(V_{\mathbb{C}}) = J_a^{2n+1}(V_{\mathbb{C}})$. Theorem 2.1 implies that the complex abelian variety $J^{2n+1}(X_{\mathbb{C}})$ has a distinguished model over K. $\square$

6. Albaneses of Hilbert schemes

Over the complex numbers the image of the Abel–Jacobi map is dominated by Albaneses of resolutions of singularities of products of irreducible components of Hilbert schemes. Since Hilbert schemes are functorial, and in particular defined over the field of definition, and since the image of the Abel–Jacobi map descends to the field of definition, one might expect this abelian variety to be dominated by Albaneses of resolutions of singularities of products of irreducible components of Hilbert schemes defined over the field of definition. In this section, we show this is the case, thereby proving Theorem B. Our approach utilizes the theory of Galois equivariant regular homomorphisms, and consequently, we obtain some related results over perfect fields in arbitrary characteristic.

Throughout this section, K denotes a perfect field and $\overline{K}$ denotes an algebraic closure of K.

6.1. Regular homomorphisms and difference maps.

In this section we give an equivalent theory of regular homomorphisms and algebraic representatives that does not rely on pointed varieties.

Let $X/\overline{K}$ be a smooth projective variety over the algebraically closed field $\overline{K}$, let $T/\overline{K}$ be a smooth integral variety and let Z be a codimension-i cycle on $T \times_{\overline{K}} X$. Let $p_{13}, p_{23} : T \times_{\overline{K}} T \times_{\overline{K}} X \to T \times_{\overline{K}} X$ be the obvious projections. Let $\widetilde{Z}$ be defined as the cycle

$$\widetilde{Z} := p_{13}^* Z - p_{23}^* Z$$

on $T \times_{\overline{K}} T \times_{\overline{K}} X$. For points $t_1, t_0 \in T(\overline{K})$, we have $\widetilde{Z}_{(t_1,t_0)} = Z_{t_1} - Z_{t_0}$. We therefore have a map

$$(T \times_{\overline{K}} T)(\overline{K}) \xrightarrow{y_Z} \mathrm{A}^i(X) \tag{6.1}$$
$$(t_1, t_0) \longmapsto Z_{t_1} - Z_{t_0}.$$

Lemma 6.1. *Let $X/\overline{K}$ be a smooth projective variety over an algebraically closed field $\overline{K}$, and let $A/\overline{K}$ be an abelian variety. A homomorphism of groups $\phi : \mathrm{A}^i(X) \to A(\overline{K})$ is regular if and only if for every pair (T, Z) with T a smooth integral variety over $\overline{K}$ and $Z \in \mathrm{CH}^i(T \times_{\overline{K}} X)$, the composition*

$$(T \times_{\overline{K}} T)(\overline{K}) \xrightarrow{y_Z} \mathrm{A}^i(X) \xrightarrow{\phi} A(\overline{K})$$

is induced by a morphism of varieties $\xi_Z : T \times_{\overline{K}} T \to A$.

Proof. If $\phi : \mathrm{A}^i(X) \to A(\overline{K})$ is a regular homomorphism to an abelian variety, then $\phi \circ y_Z$ is induced by a morphism of varieties $T \times_{\overline{K}} T \to A$; indeed after choosing any diagonal base point $(t_0, t_0) \in (T \times_{\overline{K}} T)(\overline{K})$, the maps $\phi \circ y_Z$ and $\phi \circ w_{\widetilde{Z},(t_0,t_0)}$ agree. Conversely, suppose $\phi \circ y_Z$ is induced by a morphism ξ_Z of varieties, and let $t_0 \in T(\overline{K})$ be any base point. Let ι be the inclusion $\iota : T \to T \times \{t_0\} \subset T \times T$. Then $w_{Z,t_0} = y_Z|_{\iota(T)}$, and $\phi \circ w_Z$ is induced by the morphism $\xi_Z \circ \iota$. ☐

Now suppose that X is a smooth projective variety over K, that T is a smooth integral quasi-projective variety over K, and that Z is a codimension-i cycle on $T \times_K X$. The cycle $\widetilde{Z} = p_{13}^* Z - p_{23}^* Z$ on $T \times_K T \times_K X$ is again defined over K.

Lemma 6.2. *Suppose X, Z and T are as above, and let A/K be an abelian variety. If $\phi : \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ is a $\mathrm{Gal}(K)$-equivariant regular homomorphism, then the induced morphism $\xi_{Z_{\overline{K}}} : (T \times_K T)_{\overline{K}} \to A_{\overline{K}}$ is also $\mathrm{Gal}(K)$-equivariant on $\overline{K}$-points, and thus $\xi_{Z_{\overline{K}}}$ descends to a morphism $\xi_Z : T \times_K T \to A$ of varieties over K.*

Proof. For each $\sigma \in \mathrm{Gal}(K)$ there is a commutative diagram (see [ACMV16a, Rem. 4.3])

$$\begin{tikzcd}
(T \times_K T)(\overline{K}) \arrow[r, "y_{Z_{\overline{K}}}"] \arrow[d, "\sigma^*_{T\times T}"] & \mathrm{A}^i(X_{\overline{K}}) \arrow[r, "\phi"] \arrow[d, "\sigma^*_X"] & A(\overline{K}) \arrow[d, "\sigma^*_A"] \\
(T \times_K T)(\overline{K}) \arrow[r, "y_{Z_{\overline{K}}}"] & \mathrm{A}^i(X_{\overline{K}}) \arrow[r, "\phi"] & A(\overline{K}).
\end{tikzcd}$$

Now ϕ is $\mathrm{Gal}(K)$-equivariant by hypothesis, and $y_{Z_{\overline{K}}}$ is $\mathrm{Gal}(K)$-equivariant since $\widetilde{Z}$, T and X are defined over K. Consequently, $\xi_{Z_{\overline{K}}}$ is $\mathrm{Gal}(K)$-equivariant, as claimed. ☐

6.2. **Albaneses of Hilbert schemes and the Abel–Jacobi map.** We are now in a position to prove the following theorem, which will allow us to prove Theorem B.

Theorem 6.3. *Suppose X is a smooth projective variety over a perfect field K, and let n be a nonnegative integer. Let A/K be an abelian variety defined over K, and let*

$$\phi : \mathrm{A}^{n+1}(X_{\overline{K}}) \twoheadrightarrow A_{\overline{K}}(\overline{K})$$

be a surjective Galois-equivariant regular homomorphism. Then there are a finite number of irreducible components of the Hilbert scheme $\mathrm{Hilb}^{n+1}_{X/K}$ parameterizing codimension $n+1$ subschemes of X/K, so that by taking a finite product H of these components, and then denoting by $\mathrm{Alb}_{\widetilde{H}/K}$ the Albanese variety of a smooth alteration $\widetilde{H}$ of H, there is a surjective morphism

$$\mathrm{Alb}_{\widetilde{H}/K} \twoheadrightarrow A \tag{6.2}$$

of abelian varieties over K.

Proof. Let Z be the cycle on $A \times_K X$ from [ACMV16a, Lem. 4.9(d)] so that the composition

$$A(\overline{K}) \xrightarrow{w_{\overline{Z}}} \mathrm{A}^{n+1}(X_{\overline{K}}) \xrightarrow{\phi} A(\overline{K})$$

is induced by the K-morphism $r \cdot \mathrm{Id} : A \to A$ for some positive integer r.

Now using Bertini's theorem, let C be a smooth projective curve that is a linear section of A passing through the origin (so it has a K-point), and such that the inclusion $C \hookrightarrow A$ induces a surjective morphism $J_{C/K} \twoheadrightarrow A$. Denote again by Z the refined Gysin restriction of the cycle Z to C. We have a commutative diagram

$$\begin{tikzcd} C(\overline{K}) \arrow[r, hook] \arrow[d] & A(\overline{K}) \arrow[r, "w_{\overline{Z}}"] \arrow[rr, bend right, two heads, "r\cdot\mathrm{Id}"'] & \mathrm{A}^{n+1}(X_{\overline{K}}) \arrow[r, two heads, "\phi"] & A(\overline{K}) \\ J_{C/K}(\overline{K}) \arrow[ur, two heads] & & & \end{tikzcd} \tag{6.3}$$

Discarding extra components, we may assume that Z is flat over C. Write $Z = \sum_{j=1}^{m} V^{(j)} - \sum_{j=m+1}^{m'} V^{(j)}$, where $V^{(1)}, \ldots, V^{(m')}$ are integral components of the support of Z, which by assumption are flat over C. Let $\mathrm{Hilb}^{(j)}_{X/K}$ be the component of the Hilbert scheme, with universal subscheme $U^{(j)} \subseteq \mathrm{Hilb}^{(j)}_{X/K} \times_K X$ such that $V^{(j)}$ is obtained by pull-back via a morphism $f^{(j)} : C \to \mathrm{Hilb}^{(j)}_{X/K}$. Let $H = \prod_{j=1}^{m'} \mathrm{Hilb}^{(j)}_{X/K}$, and let $U_H := \sum_{j=1}^{m} \mathrm{pr}_j^* U^{(j)} - \sum_{j=m+1}^{m'} \mathrm{pr}_j^* U^{(j)}$. There is an induced morphism $f : C \to H$ and we have $Z = f^* U_H$; the pull-back is defined since all the cycles are flat over the base.

Now let $\nu : \widetilde{H} \to H$ be a smooth alteration of H and let $\widetilde{U} = \nu^* U_H$. Let $\mu : \widetilde{C} \to C$ be an alteration such that there is a commutative diagram

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\tilde{f}"] \arrow[d, "\mu"'] & \widetilde{H} \arrow[d, "\nu"] \\ C \arrow[r, "f"] & H. \end{tikzcd}$$

Let $\widetilde{Z} = \mu^* Z$. We obtain maps

$$(\widetilde{C} \times_K \widetilde{C})(\overline{K}) \hookrightarrow (\widetilde{H} \times_K \widetilde{H})(\overline{K}) \xrightarrow{y_{\widetilde{U}_{\overline{K}}}} \mathrm{A}^{n+1}(X_{\overline{K}}) \xrightarrow{\phi} \!\!\!\!\!\rightarrow A(\overline{K}).$$

By Lemma 6.2, these descend to K-morphisms

$$\widetilde{C} \times_K \widetilde{C} \hookrightarrow \widetilde{H} \times_K \widetilde{H} \xrightarrow{\xi_{\widetilde{U}}} A.$$

Recall that if W/K is any variety, then there exist an abelian variety $\mathrm{Alb}_{W/K}$ and a torsor $\mathrm{Alb}^1_{W/K}$ under $\mathrm{Alb}_{W/K}$, equipped with a morphism $W \to \mathrm{Alb}^1_{W/K}$ which is universal for morphisms from

W to abelian torsors. Taking Albanese torsors we obtain a commutative diagram

$$\begin{tikzcd}
\widetilde{C} \times_K \widetilde{C} \arrow[r, hook] \arrow[d] \arrow[ddd, bend right=60] & \widetilde{H} \times_K \widetilde{H} \arrow[r, two heads, "\xi_{\widetilde{U}}"] \arrow[d] & A \arrow[ddd, equal] \\
\mathrm{Alb}^1_{\widetilde{C}/K} \times_K \mathrm{Alb}^1_{\widetilde{C}/K} \arrow[r] \arrow[d, two heads] & \mathrm{Alb}^1_{\widetilde{H}/K} \times_K \mathrm{Alb}^1_{\widetilde{H}/K} \arrow[ur] & \\
J_{C/K} \times_K J_{C/K} \arrow[ddrr, two heads] & & \\
C \times_K C \arrow[u] \arrow[rr, two heads, "\xi_Z"'] & & A.
\end{tikzcd}$$

The surjectivity of the map $J_{C/K} \times_K J_{C/K} \to A$ follows from (6.3). A diagram chase then shows that the map $\mathrm{Alb}^1_{\widetilde{H}/K} \times_K \mathrm{Alb}^1_{\widetilde{H}/K} \to A$ is surjective. In general, if T is a torsor under an abelian variety B/K, and if $T \twoheadrightarrow A'$ is a surjection to an abelian variety, then there is a surjection $B \to A'$ over K. (Indeed, the surjection $T \twoheadrightarrow A'$ induces an inclusion $\mathrm{Pic}^0_{A'/K} \hookrightarrow \mathrm{Pic}^0_{T/K}$; but $\mathrm{Pic}^0_{A'/K}$ is isogenous to A', while $\mathrm{Pic}^0_{T/K}$ is isogenous to B.) Applying this to the surjection $\mathrm{Alb}^1_{\widetilde{H}/K} \times_K \mathrm{Alb}^1_{\widetilde{H}/K} \twoheadrightarrow A$, we obtain the surjection $\mathrm{Alb}_{\widetilde{H}/K} \times_K \mathrm{Alb}_{\widetilde{H}/K} \to A$. Theorem 6.3 now follows, where the $\widetilde{H}$ in (6.2) is the product $\widetilde{H} \times_K \widetilde{H}$ considered here. $\square$

We now use Theorem 6.3 to prove Theorem B.

Proof of Theorem B. Recall that the Abel–Jacobi map $AJ : \mathrm{A}^{n+1}(X_{\mathbb{C}}) \longrightarrow J_a^{2n+1}(X_{\mathbb{C}})$ is a surjective regular homomorphism. By Theorem A and its proof, $J_a^{2n+1}(X_{\mathbb{C}})$ descends uniquely to an abelian variety J/K such that the surjective regular homomorphism $\underline{\underline{AJ}} : \mathrm{A}^{n+1}(X_{\overline{K}}) \longrightarrow J_{\overline{K}}$ defined in the proof of Lemma 3.3 is Galois-equivariant. Now employ Theorem 6.3. $\square$

Proof of Corollary C. A uniruled threefold has Chow group of zero-cycles supported on a surface. A decomposition of the diagonal argument [BS83] shows that the threefold has geometric coniveau 1 in degree 3. $\square$

Theorem 6.3 also gives the following result for algebraic representatives.

Corollary 6.4. *Let X be a smooth projective variety over a perfect field K, let $\Omega/\overline{K}$ be an algebraically closed field extension, with either $\Omega = \overline{K}$ or $\mathrm{char}(K) = 0$, and let n be a nonnegative integer. Assume there is an algebraic representative $\phi^{n+1}_{\Omega} : \mathrm{A}^{n+1}(X_\Omega) \to \mathrm{Ab}^{n+1}(X_\Omega)(\Omega)$ (e.g., $n = 0$, 1, or $\dim X - 1$).*

Then the abelian variety $\mathrm{Ab}^{n+1}(X_\Omega)$ descends to an abelian variety $\underline{\mathrm{Ab}}^{n+1}(X_{\overline{K}})$ over K, and there are a finite number of irreducible components of the Hilbert scheme $\mathrm{Hilb}^{n+1}_{X/K}$ parameterizing codimension $n+1$ subschemes of X/K, so that by taking a finite product H of these components, and then denoting by $\mathrm{Alb}_{\widetilde{H}/K}$ the Albanese of a smooth alteration $\widetilde{H}$ of H, there is a surjective morphism $\mathrm{Alb}_{\widetilde{H}/K} \twoheadrightarrow \underline{\mathrm{Ab}}^{n+1}(X_{\overline{K}})$ of abelian varieties over K.

Proof. The fact that $\mathrm{Ab}^{n+1}(X_\Omega)$ descends to $\overline{K}$ to give $\mathrm{Ab}^{n+1}(X_{\overline{K}})$ is [ACMV16a, Thm. 3.7]. It is then shown in [ACMV16a, Thm. 4.4] that $\mathrm{Ab}^{n+1}(X_{\overline{K}})$ descends to an abelian variety over K and that the map $\phi^{n+1}_{\overline{K}} : \mathrm{A}^{n+1}(X_{\overline{K}}) \to \mathrm{Ab}^{n+1}(X_{\overline{K}})(\overline{K})$ is $\mathrm{Gal}(K)$-equivariant. Therefore, we may employ Theorem 6.3 to conclude. $\square$

Appendix A. Cohomology of Jacobians of curves via Abel maps

Let C be a smooth projective curve over a field K. For any n invertible in K, the Kummer sequence of étale sheaves on C :

$$1 \longrightarrow \boldsymbol{\mu}_n \longrightarrow \mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m \longrightarrow 1$$

gives an isomorphism

$$H^1(\overline{C}, \boldsymbol{\mu}_n) \cong \mathrm{Pic}_{C/K}[n] = \mathrm{Pic}^{\circ}_{C/K}[n],$$

where we have written $\overline{C}$ for $C_{\overline{K}}$. After taking the inverse limit over all powers of a fixed prime $n = \ell$, we obtain isomorphisms of $\mathrm{Gal}(K)$-representations

$$H^1(\overline{C}, \mathbb{Z}_\ell) \cong T_\ell \, \mathrm{Pic}^{\circ}_{C/K} \cong H^1(\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}, \mathbb{Z}_\ell(1))^\vee.$$

Moreover, the canonical (principal) polarization on the Jacobian gives an isomorphism

$$H^1(\overline{C}, \mathbb{Z}_\ell) \cong H^1(\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}, \mathbb{Z}_\ell). \tag{A.1}$$

In this appendix we show that, up to tensoring with $\mathbb{Q}_\ell$, the isomorphism (A.1) is induced by a K-morphism $C \to \mathrm{Pic}^{\circ}_{C/K}$.

Proposition A.1. *Let C be a smooth projective curve over a field K. Then there exists a morphism $\beta : C \to \mathrm{Pic}^{\circ}_{C/K}$ over K which induces an isomorphism*

$$\beta^* : H^1(\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^1(\overline{C}, \mathbb{Z}_\ell)$$

of $\mathrm{Gal}(K)$-representations for all but finitely many ℓ. For all ℓ invertible in K, we have that the pull-back $\beta^* : H^1(\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}, \mathbb{Q}_\ell) \to H^1(\overline{C}, \mathbb{Q}_\ell)$ is an isomorphism.

The case of an integral curve over an algebraically closed field $K = \overline{K}$ is standard (e.g., [Mil08, Prop. 9.1,p.113]). The case where C is geometrically irreducible and $C(K)$ is nonempty is certainly well-known ; even if C admits no K-points, the result follows almost immediately from the case $K = \overline{K}$:

Lemma A.2. *If C/K is geometrically irreducible, then Proposition A.1 holds for C.*

Proof. Let d be a positive integer such that C admits a line bundle L of degree d over K. Let β denote the composition

$$\beta : C \xrightarrow{a} \mathrm{Pic}^1_{C/K} \xrightarrow[\text{isogeny}]{[d]=(-)^{\otimes d}} \mathrm{Pic}^d_{C/K} \xrightarrow[\sim]{(-)\otimes L^\vee} \mathrm{Pic}^{\circ}_{C/K},$$

where $\mathrm{Pic}^e_{C/K}$ denotes the torsor under $\mathrm{Pic}^{\circ}_{C/K}$ consisting of degree e line bundles on C/K, and a is the Abel map (e.g., [Kle05, Def. 9.4.6, Rem. 9.3.9]).

We claim that if $\ell \nmid d$, then $\beta^* : H^1(\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}, \mathbb{Z}_\ell) \to H^1(\overline{C}, \mathbb{Z}_\ell)$ is an isomorphism. After passage to $\overline{K}$, we may find a line bundle M such that $M^{\otimes d} \cong L$. We have a commutative diagram

$$\begin{tikzcd}
\overline{C} \arrow[r, "a"] \arrow[dr, "a_M"'] & \mathrm{Pic}^1_{\overline{C}/\overline{K}} \arrow[r, "{[d]}"] \arrow[d, "(-)\otimes M^\vee"] & \mathrm{Pic}^d_{\overline{C}/\overline{K}} \arrow[d, "(-)\otimes L^\vee"] \\
& \mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}} \arrow[r, "{[d]}"] & \mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}
\end{tikzcd}$$

Since the diagonal arrow is the usual Abel–Jacobi embedding of $\overline{C}$ in its Jacobian, where the assertion about pull back of cohomology is well known (e.g., [Mil08, Prop. 9.1, p.113]), and the lower horizontal arrow is an isogeny of degree $d^{2g(C)}$, the commutativity of the diagram implies that β has the asserted properties. $\square$

A.1. **Components of the Picard scheme.** Now suppose that $\overline{C}$ is reducible. Continue to let $\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}$ denote the connected component of identity of the Picard scheme, and for each d let $\mathrm{Pic}^{d}_{\overline{C}/\overline{K}}$ be the space of line bundles of total degree d. (This has the unfortunate notational side effect that $\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}$ does not coincide with $\mathrm{Pic}^{0}_{\overline{C}/\overline{K}}$, but we will never have cause to study the space of line bundles of total degree zero.) Then $\mathrm{Pic}^{d}_{\overline{C}/\overline{K}}$ is no longer a torsor under $\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}$, and we need to work slightly harder to identify suitable geometrically irreducible, K-rational components of the Picard scheme of C.

Let $\Pi_0(\overline{C})$ be the set of (geometrically) irreducible components of $\overline{C}$. Fix a component $\overline{D} \in \Pi_0(C_{\overline{K}})$, and let $H \subset \mathrm{Gal}(K)$ be its stabilizer. Since C is irreducible, we have

$$\overline{C} = \bigsqcup_{[\sigma]\in \mathrm{Gal}(K)/H} \overline{D}^{\sigma},$$

where viewing σ as an automorphism of $\overline{C}$, we set $\overline{D}^{\sigma} = \sigma(\overline{D})$. Let $e = \#\Pi_0(\overline{C})$. Inside the de-th symmetric power $S^{de}(C)_{\overline{K}} = S^{de}(\overline{C})$ we identify the irreducible component

$$S^{\Delta_d}(\overline{C}) := \prod_{[\sigma]\in \mathrm{Gal}(K)/H} S^{d}(\overline{D}^{\sigma}).$$

Since this element of $\Pi_0(S^{de}(\overline{C}))$ is fixed by $\mathrm{Gal}(K)$, it descends to K as a geometrically irreducible variety.

Similarly, inside the Picard scheme $\mathrm{Pic}_{\overline{C}/\overline{K}}$ we single out

$$\mathrm{Pic}^{\Delta_d}_{\overline{C}/\overline{K}} = \prod_{[\sigma]\in \mathrm{Gal}(K)/H} \mathrm{Pic}^{d}_{\overline{D}^{\sigma}/\overline{K}}.$$

It is visibly irreducible and, since it is stable under $\mathrm{Gal}(K)$, it descends to K. Note that $\mathrm{Pic}^{\Delta_d}_{\overline{C}/\overline{K}}$ is a $\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}$-torsor.

The (de)-th Abel map $S^{de}(C) \to \mathrm{Pic}^{de}_{C/K}$ then restricts to a morphism

$$S^{\Delta_d}(C) \xrightarrow{a_{\Delta_d}} \mathrm{Pic}^{\Delta_d}_{C/K}$$

of geometrically irreducible varieties over K.

One (still) has the canonical Abel map

$$C \xrightarrow{a} \mathrm{Pic}^{1}_{C/K}.$$

Over $\overline{K}$, the image of the Abel map $a_{\overline{K}}$ lands in

$$\mathrm{Pic}^{\mathbf{1}}_{\overline{C}/\overline{K}} = \bigsqcup_{[\sigma]\in \mathrm{Gal}(K)/H} \left(\mathrm{Pic}^{1}_{\overline{D}^{\sigma}} \times \prod_{[\tau]\neq[\sigma]} \mathrm{Pic}^{\circ}_{\overline{D}^{\tau}} \right).$$

Although $\mathrm{Pic}^{\mathbf{1}}_{\overline{C}/\overline{K}}$ has e components, $\mathrm{Gal}(K)$ acts transitively on them, and we have an irreducible variety $\mathrm{Pic}^{\mathbf{1}}_{C/K}$ over K.

In conclusion, the canonical Abel map induces a morphism

$$C \xrightarrow{a} \mathrm{Pic}^{\mathbf{1}}_{C/K}$$

of irreducible varieties over K.

We need two more K-rational morphisms:

Lemma A.3. *Let C/K be a smooth projective integral curve. Let s be the map*

$$\begin{aligned}\mathrm{Pic}^{\mathbf{1}}_{\overline{C}/\overline{K}} &\xrightarrow{\quad s \quad} \mathrm{Pic}^{\Delta_1}_{\overline{C}/\overline{K}} \\ L &\longmapsto \textstyle\bigotimes_{[\sigma]\in \mathrm{Gal}(K)/H} \sigma^* L.\end{aligned}$$

Let t be the map

$$\overline{C} \xrightarrow{\ t\ } S^{\Delta_1}(\overline{C})$$

such that, if $P \in \overline{D}^{\tau}(K) \subset C(\overline{K})$, then the components of $t(P)$ are given by

$$t(P)_\sigma = \sigma\tau^{-1}(P) \in \overline{D}^{\sigma}$$

Then s and t descend to morphisms over K.

Proof. Each is $\mathrm{Gal}(K)$-equivariant on $\overline{K}$-points. ☐

A.2. **Isomorphisms on cohomology.**

Lemma A.4. *Let C/K be a smooth projective irreducible curve. Then the composition*

$$C \xrightarrow{\ a\ } \mathrm{Pic}^{\mathbf{1}}_{C/K} \xrightarrow{\ s\ } \mathrm{Pic}^{\Delta_1}_{C/K}$$

induces an isomorphism of $\mathrm{Gal}(K)$*-representations*

$$H^1(\mathrm{Pic}^{\Delta_1}_{\overline{C}/\overline{K}}\mathbb{Z}_\ell) \to H^1(\overline{C}, \mathbb{Z}_\ell).$$

Proof. It suffices to analyze $s \circ a$ after base change to $\overline{K}$. Choose a base point $P_\sigma \in \overline{D}^{\sigma}$ for each irreducible component of $\overline{C}$. We have a commutative diagram

$$\begin{tikzcd} \overline{C} \arrow[r, "a"] \arrow[d, "t"] & \mathrm{Pic}^{\mathbf{1}}_{\overline{C}/\overline{K}} \arrow[r, "s"] & \mathrm{Pic}^{\Delta_1}_{\overline{C}/\overline{K}} \arrow[d, "\prod_{[\sigma]}(-)\otimes\mathcal{O}(-P_\sigma)"] \\ S^{\Delta_1}(\overline{C}) \arrow[urr, "a_{\Delta_1}"] \arrow[rr, "\prod_{[\sigma]} a_{P_\sigma}"'] & & \mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}, \end{tikzcd}$$

where the bottom arrow is the product of Abel maps associated to the points P_σ. Since the right-most vertical arrow is an isomorphism of schemes, it suffices to verify that t and $\prod a_{P_\sigma}$ induce isomorphisms on first cohomology groups. On one hand, since cohomology takes coproducts to products, we have $H^1(\overline{C}, \mathbb{Z}_\ell) \cong \prod_\sigma H^1(\overline{D}^{\sigma}, \mathbb{Z}_\ell)$. On the other hand, since each $\overline{D}^{\sigma}$ is connected, the Künneth formula implies that $H^1(S^{\Delta_1}(\overline{C}), \mathbb{Z}_\ell) = H^1(\prod_\sigma \overline{D}^{\sigma}, \mathbb{Z}_\ell) \cong \oplus_\sigma \mathrm{pr}_\sigma^* H^1(\overline{D}^{\sigma}, \mathbb{Z}_\ell)$. Since the composition $\overline{D}^{\tau} \xrightarrow{\ t\ } \prod_\sigma \overline{D}^{\sigma} \xrightarrow{\mathrm{pr}_\tau} \overline{D}^{\tau}$ is the identity,

$$H^1(S^{\Delta_1}(\overline{C}), \mathbb{Z}_\ell) \xrightarrow{\ t^*\ } H^1(\overline{C}, \mathbb{Z}_\ell)$$

is an isomorphism as well.

Finally, since each Abel–Jacobi map a_{P_σ} induces an isomorphism $H^1(\mathrm{Pic}^{\circ}_{\overline{D}^{\sigma}/\overline{K}}, \mathbb{Z}_\ell) \cong H^1(\overline{D}^{\sigma}, \mathbb{Z}_\ell)$, their product yields an isomorphism $(\prod_{[\sigma]} a_{P_\sigma})^* : H^1(\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}, \mathbb{Z}_\ell) \to H^1(S^{\Delta_1}(\overline{C}), \mathbb{Z}_\ell)$. ☐

It is now straight-forward to provide a proof of the main result of this appendix.

Proof of Proposition A.1. Since both the Picard functor and cohomology take coproducts to products, we may and do assume that C is irreducible. Choose d such that $\mathrm{Pic}^{\Delta_d}_{C/K}$ admits a K-point L. Let β be the composition

$$C \xrightarrow{a} \mathrm{Pic}^{\mathbf{1}}_{C/K} \xrightarrow{s} \mathrm{Pic}^{\Delta_1}_{C/K} \xrightarrow[\text{isog.}]{[d]} \mathrm{Pic}^{\Delta_d}_{C/K} \xrightarrow[\cong]{(-)\otimes L^\vee} \mathrm{Pic}^{\circ}_{C/K}\,.$$

By Lemma A.4, $\beta^* : H^1(\mathrm{Pic}^{\circ}_{\overline{C}/\overline{K}}, \mathbb{Z}_\ell) \to H^1(\overline{C}, \mathbb{Z}_\ell)$ is an isomorphism as long as $\ell \nmid d$. $\square$

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