Sept 4

Note Title 9/4/2009

$$D \subseteq \mathbb{R}^n$$

$$\vec{f} : D \to \mathbb{R}^m$$

$$\vec{f}(\vec{x}) = (f_1(\vec{x}), \cdots, f_m(\vec{x}))$$

$$\vec{a} \in \overline{D}$$

$$\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$$

if $\forall \varepsilon > 0 \quad \exists \delta > 0$

& if $0 < \|\vec{x} - \vec{a}\| < \delta$ $\longleftarrow$

then $\|\vec{f}(\vec{x}) - \vec{L}\| < \varepsilon.$ $\longleftarrow$

$$\left[\begin{array}{c} 0 < \|\vec{x} - \vec{a}\| < \delta \\ \iff \\ \vec{x} \neq \vec{a} \ \& \ \vec{x} \in B_\delta(\vec{a}) \end{array} \right]$$

$$\left[\begin{array}{c} \|\vec{f}(\vec{x}) - \vec{L}\| < \varepsilon \\ \iff \\ \vec{f}(\vec{x}) \in B_\varepsilon(\vec{L}) \end{array} \right]$$

Lemma If $\vec{L} = (L_1 \dots L_m)$

then $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$.

iff

$\forall i = 1, \dots, m$

$\lim_{\vec{x} \to \vec{a}} f_i(\vec{x}) = L_i$.

Pf

in vector valued limit have

" $\|\vec{f}(\vec{x}) - \vec{L}\| < \varepsilon$ "

in scalar limits

" $|f_i(\vec{x}) - L_i| < \varepsilon$ "

$$\max(|f_i(\vec{x}) - L_i|) \leq \|\vec{f}(\vec{x}) - \vec{L}\| \leq \sqrt{m} \max(|f_i(\vec{x}) - L_i|)$$

Def $f : D \to \mathbb{R}^m$

We say $\vec{f}$ is continuous at $\vec{a} \in D$ if $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{f}(\vec{a})$.

Means $\forall \varepsilon > 0 \ \exists \delta > 0$ so that if

$$\cancel{0 <} \|\vec{x} - \vec{a}\| < \delta$$

then

$$\|f(\vec{x}) - f(\vec{a})\| < \varepsilon.$$

<u>Cor</u> $\vec{f}$ is continuous at $\vec{a}$ iff each f_i, $i = 1, \dots, m$ is cont.

<u>Thm</u> Suppose $D \subseteq \mathbb{R}^n$,

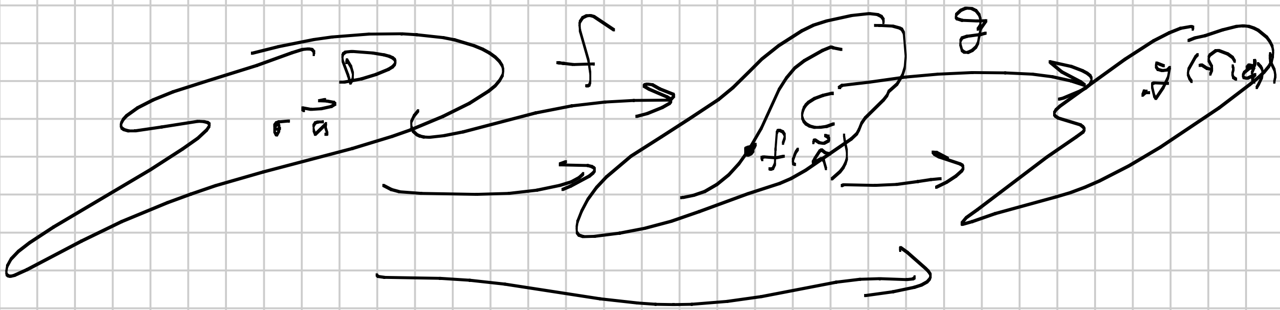
$\vec{f} : D \to \mathbb{R}^m$,

$\vec{f}(D) \subseteq C$

$\vec{g} : C \to \mathbb{R}^k$

$\vec{a} \in D$, $\vec{f}$ is cont at $\vec{a}$ & $\vec{g}$ is cont. at $\vec{f}(\vec{a})$.

Then $(\vec{g} \circ \vec{f})$ is cont. at $\vec{a}$.



pf. Let $\varepsilon > 0$. There is then a $\delta_1 > 0$ & if $\|\vec{y} - \vec{f}(\vec{a})\| < \delta_1$ then $\|\vec{g}(\vec{y}) - \vec{g}(\vec{f}(\vec{a}))\| < \varepsilon$.
(Cont. of $\vec{g}$ at $\vec{f}(\vec{a})$)

Given $\delta_1 > 0$, $\exists\ \delta > 0$ & if $\|\vec{x} - \vec{a}\| < \delta$ then $\|\vec{f}(\vec{x}) - \vec{f}(\vec{a})\| < \delta_1$

Suppose $\|\vec{x} - \vec{a}\| < \delta$.
$\Rightarrow \|\vec{f}(\vec{x}) - \vec{f}(\vec{a})\| < \delta_1$
$\Rightarrow \|\vec{g}(\vec{f}(\vec{x})) - \vec{g}(\vec{f}(\vec{a}))\| < \varepsilon$.

Lemma

$$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x,y) \xrightarrow{P} x+y$$

$$(x,y) \xrightarrow{T} xy$$

$$\mathbb{R}\setminus 0 \longrightarrow \mathbb{R}$$

$$x \xrightarrow{R} 1/x$$

These are cont. where defined.

Pf. of 1st

$$P(x,y) = x+y.$$

Cont. at (a,b)

Given $\varepsilon > 0$, let $\delta = \varepsilon/2$

$$\max(|a-x|, |b-y|) \leq \|(x,y)-(a,b)\| < \delta$$

$$|x-a| < \varepsilon/2 \ \&$$

$$|y-b| < \varepsilon/2.$$

So

$$|P(x,y) - P(a,b)|$$

$$= |(x+y) - (a+b)|$$

$$= |(x-a)+(y-b)|$$
$$\leq |x-a| + |y-b| < \varepsilon.$$

Thm $D \subseteq \mathbb{R}^n$,

$f, g : D \to \mathbb{R}$.

& $f+g$ are cont. at $\vec{a}$.

then

1) $(f+g)$ is cont at $\vec{a}$

2) (fg) is cont at $\vec{a}$

3) if $f(\vec{a}) \neq 0$,

$1/f$ is cont at $\vec{a}$.

Pf

$$D \ni \vec{x} \xrightarrow{F} (f(\vec{x}), g(\vec{x})) \in \mathbb{R}^2$$

F is cont. at $\vec{a}$.

$$\mathbb{R}^2 \xrightarrow{P} \mathbb{R}, \quad f(\vec{x}) + g(\vec{x})$$

$$\xrightarrow{m} f(\vec{x})\, g(\vec{x})$$

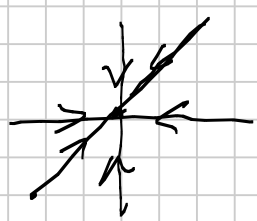
$$\vec{x} \longrightarrow f(\vec{x}) \xrightarrow{R} \frac{1}{f(\vec{x})}$$

Cor Polynomials are continuous.

Rat'l fctns away from roots of denominator

Ex $f(x,y) = \dfrac{x^2 y}{x^4+y^2}$, 0 at $\vec{0}$

Cont away from $(0,0)$.



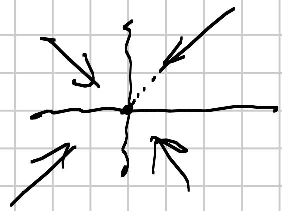
$y=0$, f - vanishes
$x=0$ " "

$$y = cx \qquad c \neq 0$$

$$f(x,y) = \frac{cx^3}{x^4+c^2x^2}$$

$$= \frac{cx}{x^2+c^2}$$

$$\lim_{x\to 0} \frac{cx}{x^2+c^2} = 0$$



Error

$$f(x,y) = \frac{cx}{x^2+c^2}$$

What is max & where.

$$\left(\frac{cx}{x^2+c^2}\right)' = \frac{c(x^2+c^2)-2cx^2}{(x^2+c^2)^2}$$

$$\frac{c^3 - cx^2}{\cancel{(x^2+c^2)^2}} = 0$$

Error was here, had c^2 & should be c^3.

$$c^2 = x^2$$

$$x = c$$

Value at c is

$$\frac{c^2}{2c^2} = \frac{1}{2}$$

as $c \to 0$ the point where max occurs

is $(c, c^2) \to \vec{0}$

& $f(c, c^2) = \frac{1}{2}$

does not $\to 0$.