Note Title Sept 18 9/18/2009

Compact Sets

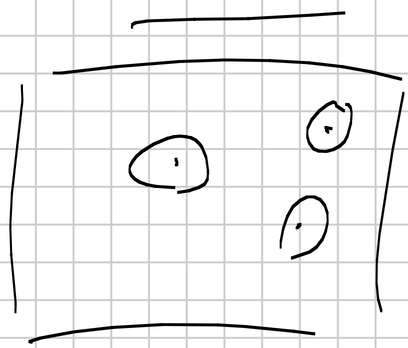
$C \subseteq \mathbb{R}^n$

an "open cover" for C is a collection $\mathcal{U}$ of of open sets with $C \subseteq \bigcup_{u \in \mathcal{U}} u$.

A subcover $\mathcal{V}$ is a subset of $\mathcal{U}$ that still covers C.

A "finite subcover" is one that has only finitely many elements.

Def A set C is called compact if every open cover of C has a finite subcover.



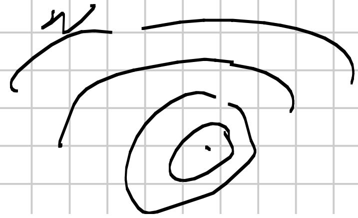
Lemma

a) If C is cpct then C is bounded

b) If C is cpct then C is closed.

pf C - unbounded.

$U_n = B_n(\vec{0})$.



$$\bigcup_{N=1}^{\infty} U_N = \mathbb{R}^n.$$

<u>Cover</u>. If

$U_{N_1}, U_{N_2} \cdots U_{N_k}$

cover C.

$$R = \max(N_1, \dots N_k)$$

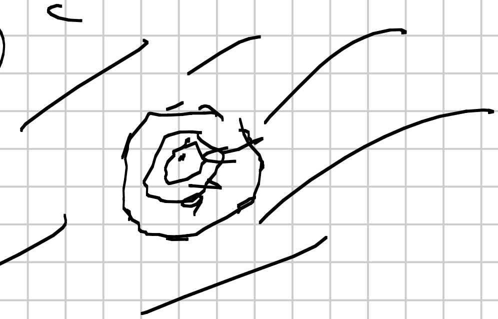
$$C \subseteq B_R(\vec{0})$$

$\Rightarrow$ <u>C - bded.</u>

b) C is <u>not</u> closed.

so $\exists\ \vec{a} \in \partial C,\ \vec{a} \notin C$.

$$U_n = \left(\overline{B_{1/n}(\vec{a})}\right)^c$$



$$\bigcup_{n=1}^{\infty} U_n = \mathbb{R}^n - \{\vec{a}\}.$$

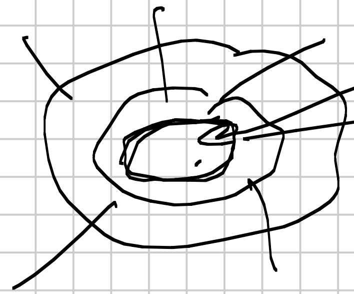
a cover. If $\exists$

$U_{n_1} \cdots U_{n_k}$ cover

C, then

let $r = \min(\frac{1}{n_1}, \dots, \frac{1}{n_k})$

& $B_r(\vec{a}) \cap C = \emptyset$



no pts of C.

$\Rightarrow \vec{a} \notin \partial(C)$.

$\Rightarrow\Leftarrow$

__Thm__ If $C \subseteq \mathbb{R}^n$ is closed & bdd then it is cpct.

(__Heine-Borel__ Thm)

__Def__ We say C is "sequentially cpct" if any sequence a_i with values in C has a conv. subsequ. $a_{k_i} \to \vec{a} \in C$.

<u>Thm</u> $C \subseteq \mathbb{R}^n$ is sequ. cpct. iff it is closed + bded.

<u>pf</u> a) C - unbded $\Rightarrow$ not sequ. cpct.

Choose a_i, $\|a_i\| > i$

b) C - not closed $\Rightarrow$ C - not sequ. cpct.

$\exists\ \vec{a} \in \partial(C)$, $\vec{a} \notin C$.

$\vec{a} \in \partial(C) \Rightarrow \exists\ \vec{a}_i \in C$, $\vec{a}_i \to \vec{a}$.

as $\vec{a} \in C$, no subs. of the $\vec{a}_i$'s can conv. to a pt of C – they all conv. to $\vec{a}$.

c) C – closed & bded &

$a_i \in C$; a_i has a conv. subsequ. as a_i's bded

$\vec{a}_{k_i} \to \vec{a}$,

if $\vec{a} \in C$ – done.

if $\vec{a} \notin C$ –

$\vec{a} \in \partial C \subseteq C$.

$\Rightarrow \Leftarrow$.

Thm Suppose $C \subseteq \mathbb{R}^n$ is cpct & $f : C \to \mathbb{R}^m$

is continuous.

Then $f(C)$ is cpct.

pf $f(C)$

Suppose $\mathcal{U}$ is an open cover for $f(C)$.

Let $\mathcal{V} = \{ f^{-1}(U) : U \in \mathcal{U} \}$

$\mathcal{V}$ is an open cover C.

C cpct so $\exists$

$V_1 \cdots V_k$ cover C.

$$\begin{array}{ccc} V_1 & & V_k \\ \| & & \| \\ f^{-1}(U_1) & & f^{-1}(U_k). \end{array}$$

$(\; f(f^{-1}(A)) = A)$

$\Rightarrow U_1 \cdots U_k$ cover C.

2nd pf

Show $f(C)$ is seq. cpct.

Let $a_i \in f(C)$

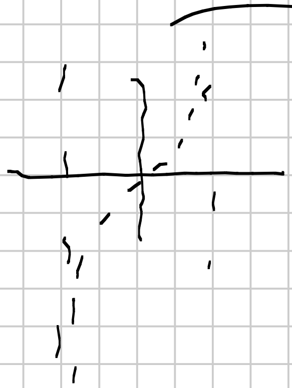
Let $b_i \in C$, $f(b_i) = c_i$.

b_i has a conv subs.

$$\vec{b}_{k_i} \to \vec{b} \in C.$$

f is cont at $\vec{b}$.

$$\vec{c}_{k_i} = f(\vec{b}_{k_i}) \to f(\vec{b}) \in f(C).$$



Thm If $C \subseteq \mathbb{R}^n$ is cpct & $f: C \to \mathbb{R}$ is cont then f achieves a max & min on C.

Pf $f(C)$ is cpct hence closed & bdd & so $f(C)$ has a max & min.