Oct 30

Note Title 10/30/2009

Thm $S \subseteq \mathbb{R}^{n+1}$ open

$F: S \to \mathbb{R}$, C^1

$(\vec{a}, b) \in S$

$F(\vec{a}, b) = 0$ +

$\partial_y F(\underline{\vec{a}, b}) \neq 0$. //

$\exists$ $r_0, r_1 > 0$,

a) For each $\vec{x} \in B_{r}(\vec{a})$ $\exists$ a unique y, $|y - b| < r$, with $F(\vec{x}, y) = 0$.

Let $f(x) = \underline{y}$.

f is <u>continuous</u>

b) f is C^1,

$$\partial_{x_j} f(\vec{x}) = - \frac{\partial_{x_j} F(\vec{x}, f(\vec{x}))}{\partial_y F(\vec{x}, f(\vec{x}))}.$$

prove b)

$$\vec{h} = h\vec{e}_j$$

$$f(\vec{x}+\vec{h}) - \underset{y}{\underset{\shortparallel}{f(\vec{x})}} = k$$

$$y + k = f(\vec{x}+\vec{h}).$$

$$F(\vec{x}+\vec{h}, y+k) = \underset{0}{\underset{\shortparallel}{F(\vec{x}+\vec{h}, f(\vec{x}+\vec{h}))}}$$

$$F(\vec{x}, y) = 0$$

$$\underline{\text{By MVT}}$$

$$0 = F(\vec{x}+\underline{\vec{h}, y+k}) - F(\vec{x}, y)$$

$$u(t) = F(\vec{x}+\underline{th}\vec{e}_j, y+\underline{t}k)$$

$$u(1) - u(0) = u'(c)$$

$$\underline{c \in (0,1)}.$$

$$u'(c) = h\,\partial_{x_j} F(x+ch\vec{e}_j, y+ck)$$
$$+\,k\,\partial_y F(x+ch\vec{e}_j, y+ck)$$

$$= 0$$

$$\frac{f(\vec{x}+h\vec{e}_j) - \overline{f(\vec{x})}}{h} = \frac{k}{h}$$

$$= - \frac{\partial_{x_j}(\vec{x}+c\underline{\vec{h}}, y+c\underline{k})}{\partial_y(\vec{x}+c\underline{\vec{h}}, y+c\underline{k})}$$

let $h \to 0$

$$\longrightarrow - \frac{\partial_{x_j}(\vec{x}, y)}{\partial_y(\vec{x}, y)}$$

Ex

$$\overbrace{\underline{\underline{a}}\,x + by}^{F} = 0$$

$$y = -\frac{a}{b}x = f(x)$$

$$a = \partial_x F, \quad b = \partial_y F$$

$$-\frac{a}{b} = -\frac{\partial_x F}{\partial_y F}$$

$$f'(x) = -\partial_x F / \partial_y F$$

Cor Suppose

$F: S \to \mathbb{R}$

$S \subseteq \mathbb{R}^n$

$\vec{a} \in S$, $F(\vec{a}) = 0$

& $\nabla F(\vec{a}) \neq 0$

Then the set of pts near $\vec{a}$ where $F(\vec{x}) = 0$ is the graph of one var. as a C^1-fctn of the others.

$\nabla F(\vec{a}) \neq 0$

$\Rightarrow \exists\, x_i,\ \partial_{x_i}(F)(\vec{a}) \neq 0$

$\Rightarrow$ x_i can be written as a C^1-fctn of the other vars.

Ex $x - y^2 - 1 = 0$

$\partial_x F = 1$

$x = y^2 + 1$

$\partial_y F = -2y$

$y \neq 0$ -

y is a fctn of x

$y^2 = x - 1$

$y = \pm\sqrt{x-1}$

if basept has $y > 0$.

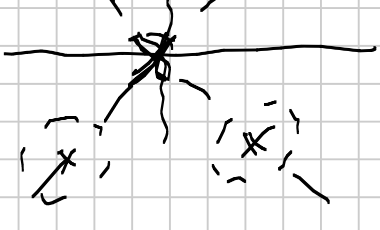
$\sqrt{x-1}$

or $y < 0$

$-\sqrt{x-1}$

Ex $F(x,y) = x^2 - y^2 = 0$

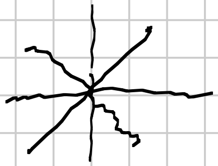
$(x-y)(x+y)$



$\partial_x F = 2x$

$\partial_y F = 2y$

$\vec{0}$



Higher dim.

$\vec{F} : S \to \mathbb{R}^k$ S - open

$S \subseteq \mathbb{R}^{n+k}$

$\mathbb{R}^{n+k}$: $(x_1, \dots, x_n, y_1, \dots y_k)$

pt $(\vec{a}, \vec{b})$

$F(\vec{a}, \vec{b}) = \vec{0}$

$$F = \begin{aligned} F_1(\vec{a}, \vec{b}) &= 0 \\ F_2(\vec{a}, \vec{b}) &= 0 \\ &\vdots \\ F_k(\vec{a}, \vec{b}) &= 0 \end{aligned}$$

Linear Case

$F(\vec{x}, \vec{y})$

$$= M \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}$$

$$\underset{k \times (n+k)}{\left[\begin{array}{c|c} A & B \end{array}\right]} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_k \end{bmatrix} = 0$$

$$\underline{\underline{A\vec{x} + B\vec{y} = 0}}$$

$$\overline{\vec{y} = -B^{-1}A\vec{x}}$$

Thm (IFT)

Suppose $F : S \to \mathbb{R}^k$

S - open in $\mathbb{R}^{n+k}$

1) $\exists\ (\vec{a}, \vec{b}) \in S,\ F(\vec{a}, \vec{b}) = 0.$

2) For $B_{i,j} = \dfrac{\partial F_i}{\partial y_j}(\vec{a}, \vec{b})$

the $k \times k$ matrix

B is invertible

There are then

#'s r_0, r_1

& for $\vec{x} \in B_{r_0}(\vec{a})$

in $B_{r_1}(\vec{b})$ there is

a unique $\vec{y}$, $F(\vec{x},\vec{y}) = 0$

Call it $f(\vec{x})$.

f is a C^1 fctn &

$$DF(\vec{x},\vec{y}) = \left[A(\vec{x},\vec{y}) \;\vdots\; B(\vec{x},\vec{y}) \right]$$

Then

$$\underline{Df = -B^{-1}A.}$$