Oct 26

Note Title 10/26/2009

HW p 99
1, a, c, e, 2

f is C^2

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \nabla f(\vec{a})\cdot\vec{h} + \frac{1}{2}\vec{h}^T H \vec{h} + R_{\vec{a},2}(\vec{h}).$$

$H = H(f)(\vec{a})$
Hessian

$$\left[\ \partial_{x_i x_j}(f)(\vec{a})\ \right]$$

Symmetric matrix.
$\Rightarrow$ H has n-real eigenvalues - eigenvectors are orthogonal + real

$\exists$ orthonormal basis of eigenvectors.

Critical Points —

Thm If S is open, f is C^1 & has a local ext. value at $\vec{a} \in S$ then $\nabla f(\vec{a}) = \vec{0}$.

pf If f has a loc. ext. value at $\vec{a}$ then for all dir $\vec{u}$,

$$g_{\vec{u}}(t) = f(\vec{a} + t\vec{u})$$

has a loc. ext. value at 0.

so $g_{\vec{u}}'(0) = 0$

$$g_{\vec{u}}'(0) = \partial_{\vec{u}} f(\vec{a})$$

$$= \nabla f(\vec{a}) \cdot \vec{u} = 0$$

$\Rightarrow \nabla f(\vec{a}) = 0$

Def.

A pt. $\vec{a} \in S$ is called a "critical pt." if $\nabla f(\vec{a}) = 0$.

- 2nd deriv test -

assume $\nabla f(\vec{a}) = \vec{0}$ + $f - C^2$

Look at

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \cancel{\nabla f(\vec{a})\cdot h} + h^T H h + R_{\vec{a},2}(\vec{h})$$

Ignore

Lets assume $\vec{u}$ is an eigenvector of H, eigenvalue λ

$$f(\vec{a}+h\vec{u}) = g_{\vec{u}}(h)$$

$$f(\vec{a} + h\vec{u}) \cong f(\vec{a}) + (h\vec{u})^T H (h\vec{u})$$

$$\cong f(\vec{a}) + h^2(\lambda \vec{u}^T \vec{u})$$

$$= f(\vec{a}) + \lambda h^2.$$

$$\cong g_{\vec{u}}(h)$$

quadratic

opens up $\lambda > 0$

down $\lambda < 0$

$\lambda = 0$ - problematic -

If $\lambda > 0$,

$g_{\vec{u}}$ has a loc min at $\vec{a}$.

$\lambda < 0$

$g_{\vec{u}}$ has a loc max at $\vec{a}$.

So 1) If H has both pos + neg

eigenvalues then $\vec{a}$ is <u>not</u> a loc. ext <u>value</u>.

2) If some eigenvalues $= 0$, then we cannot conclude <u>anything</u>.

Suppose all eigenvalues are > 0.

$\vec{u}_1, \ldots, \vec{u}_n$ - eigen v.

$\lambda_1 \ldots \lambda_n$

$$\vec{u} = \sum_{i=1}^{n} a_i \vec{u}_i$$

$$f(\vec{a} + h\vec{u}) \cong$$

$$f(\vec{a}) + h^2 \left(\sum_{j=1}^{n} a_j \vec{u}_j\right)^T H \left(\sum_{i=1}^{n} a_i \vec{u}_i\right)$$

$$= f(\vec{a}) + h^2 \left(\sum_{i,j} a_i a_j \lambda_i \vec{u}_i \cdot \vec{u}_j\right)$$

$$= f(\vec{a}) + h^2 \underbrace{\sum_{i=1}^{m} a_i^2 \lambda_i}_{> 0}$$

— Quad opening up —

The coeff
$$C_{\vec{u}} = \sum_{i=1}^{n} a_i^2 \lambda_i$$

is always > 0
& hence bounded
away from 0.

In all dir f has
a loc. min - uniformly -
& f has a loc min
at $\vec{a}$.

Symmetrically - if
all $\lambda_i < 0$.
then f has a
local max at $\vec{a}$.

Note: If some $\lambda_i < 0$
& some $\lambda_i > 0$
(none $= 0$)

We say f has a saddle at $\vec{a}$.

In 2 dim - problem easy -

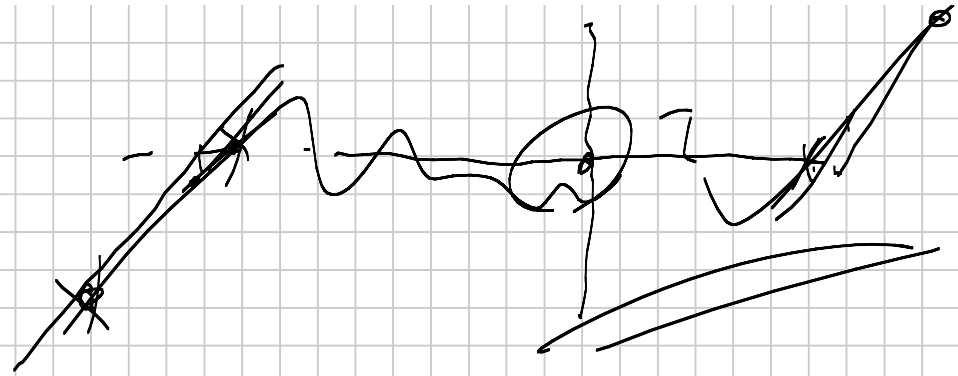
1) $\det(H) = \lambda_1 \lambda_2$

2) $\operatorname{trace}(H) = \lambda_1 + \lambda_2$

1) If $\det(H) < 0$
saddle

2) If $\det(H) > 0$
loc extval.
trace > 0 - loc min
< 0 - loc max

3) If $\det(H) = 0$
indef.



Ex $f = (x-1)(x^2-y^2)$

$\nabla f = \big(x^2 - y^2 + 2x(x-1),$
$-2y(x-1)\big)$

$= (0,0)$

$\big(3x^2 - y^2 - 2x, -2y(x-1)\big)$

either $y = 0$, or $x = 1$

$3x^2 - 2x = 0$ $\quad$ $-y^2 + 1 = 0$

$x(3x-2) = 0$ $\quad$ $y^2 = 1$

$x = 0, \ x = \frac{2}{3}$ $\quad$ $y = \pm 1$

$(0,0), (\frac{2}{3}, 0)$ $\quad$ $(1,1)$, $(1,-1)$

$\partial_{x,x} f = 6x - 2$

$\partial_{x,y} f = -2y$

$\partial_{yy} f = 2 - 2x$

$(1, 1)$, $H = \begin{bmatrix} 4 & -2 \\ -2 & 0 \end{bmatrix}$

det(H) = 4 – crit pt

trace = 4 – true min