Oct 21

Note Title 10/21/2009

$$f(a+h) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j + R_{a,k}(h)$$

If f is C^{k+1},

$$R_{a,k}(h) = \frac{h^{k+1}}{k!} \int_0^1 (1-t)^k f^{(k+1)}(a+th)\, dt$$

If f is C^k

$$R_{a,k}(h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} \left(f^{(k)}(a+th) - f^{(k)}(a) \right) dt$$

Cor If f is of Class C^k then

$$\lim_{h \to 0} \frac{R_{a,k}(h)}{h^k} = 0$$

"Error vanishes to order k."

proof $f^{(k)}$ is cont. so given $\varepsilon > 0$ $\exists \delta$ + if $|e| < \delta$ then

$$|f(a+e) - f(a)| < \varepsilon$$

Assume $h < \delta$.

$$\left| \frac{R_{a,k}(h)}{h^k} \right| = \left| \frac{1}{(k-1)!} \int_0^1 (1-t)^k \left(f^{(k)}(a+th) - f^{(k)}(a) \right) dt \right|$$

$$\leq \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} \underbrace{\left| f^{(k)}(a+th) - f^{(k)}(a) \right|}_{< \varepsilon} dt$$

$$\leq \varepsilon \frac{1}{(k-1)!} \underbrace{\int_0^1 (1-t)^{k-1} dt}_{\frac{1}{k}}$$

$$= \frac{\varepsilon}{k!} < \varepsilon \quad \checkmark$$

Lagrange form for $R_{a,k}(h)$.

Thm f is class C^{k+1} on I, $a \in I$. Let $I_0 \subseteq I$ be int. with endpts a, $a+h$.

There is a pt $c \in I_0$ s.t.

$$R_{a,k}(h) = \frac{f^{(k+1)}(c)}{(k+1)!} h^{k+1}$$

$$f(a) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j + \frac{f^{(k+1)}(c)}{(k+1)!} h^{k+1}$$

Lemma Suppose g is cont. on I_0, (closed int a to $a+h$)

Then $\exists c \in I_0$,

$$(k+1)\int_0^1 (1-t)^k g(a+ht)\,dt = g(c).$$

proof: g is cont on a closed & bdd int. so has Max, M, min, m

$t \in [0,1]$

$$M \geq g(a+ht) \geq m$$

$$M\int_0^1 (1-t)^k dt \geq \int_0^1 (1-t)^k g(a+ht) \geq m\int_0^1 (1-t)^k dt$$

$$\int_0^1 (1-t)^k dt = -\frac{(1-t)^{k+1}}{k+1}\Big|_0^1$$

$$= \frac{1}{k+1}$$

$$M \geq (k+1)\int_0^1 (1-t)^k g(a+ht)\,dt \geq m$$

$M = g(y)$, $m = g(z)$

So $\exists$ c between y & z, in I_0

$$g(c) = (k+1)\int_0^1 (1-t)^k g(a+ht)\,dt$$

Proof of Thm

$$R_{a,k}(h) = \frac{h^{k+1}}{k!}\int_0^1 (1-t)^k f^{(k+1)}(a+ht)\,dt$$

$$g(t) = f^{(k+1)}(t).$$

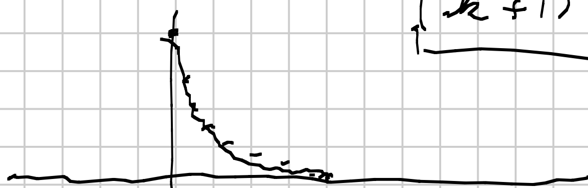
& $\exists$ $c \in I_0$

$$= \frac{h^{k+1}}{(k+1)!}\underbrace{(k+1)\int_0^1 (1-t)^k f^{(k+1)}(a+ht)\,dt}_{f^{(k+1)}(c).}$$

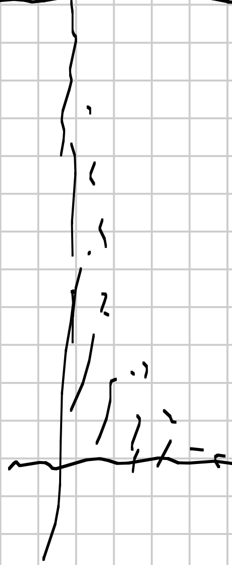
$$= \frac{f^{(k+1)}(c)}{(k+1)!}h^{k+1}$$

$$(k+1)\int_0^1 (1-t)^k g(t)\,dt$$

$$(k+1)(1-t)^k = k(t)$$



$$\int_0^1 k(t)\,g(t)\,dt$$



$$k=0,\quad \int_0^1 g(t)\,dt$$

$$\int_0^1 k(t)\,dt = 1$$

Ex $\quad f(x) = e^x$

$$\sum_{j=0} \frac{x^j}{j!}$$

$\cos x$ — 1
$-\sin$ — 0
$-\cos$ — -1
$\sin$ — 0

$$\sum_{j=0} \frac{(-1)^j x^{2j}}{(2j)!}$$

sin

$$\sum_{j=0} \frac{(-1)^j x^{2j+1}}{(2j+1)!}$$

$$f(x) = \ln(1+x)$$

$$f' = \frac{1}{1+x}$$

$$f'' = -\frac{1}{(1+x)^2}$$

$$f^{(k)} = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k}$$

$$\sum_{j=1} \frac{(-1)^{j-1} x^j}{j}$$

Operators

∂_{x_i} : functions $\to$ fcts.

linear operators

$\partial_{x_i}(f+g)$

$= \partial_{x_i} f + \partial_{x_i} g$

$(\partial_{x_i} + \partial_{x_j}) f = \partial_{x_i} f + \partial_{x_j} f.$

$(\partial_{x_i} \partial_{x_j}) f$

$= \partial_{x_i} (\partial_{x_j} f)$

Mult. Commutes

('Cause we assume cont. deriv.

$$(h_1 \partial_{x_1} + h_2 \partial_{x_2} + \cdots + h_m \partial x_m)^k$$