Oct 14

Note Title 10/14/2009

HW
p. 84 #4, 9

Higher order partials
&
Polynomial Approx.

$$P(x,y) = \boxed{2x^2y + 3xy^2 + y}$$

If I didn't know coeff.
I can find them
by diff.

$$\partial_y P(x,y) = 2x^2 + 6xy + 1.$$

$$\partial_y P(0,0) = 1$$

- coeff of y.

$$\partial_y \partial_y P(x,y) = 6x$$

$$\partial_x \partial_y \partial_y P(x,y) = 6$$

$$\partial_x \partial_y \partial_y P(0,0) = 6$$

$$\frac{6}{2} = 3 \text{ is}$$

coeff of xy^2

Higher order partial deriv.

fcts partials fcts

$$\partial_y \partial_x$$

$$\partial_{x_j} \partial_{x_i} f = \partial_{i,j} f$$

$$\partial_x \partial_x$$

pure partial

$\partial_x \partial_y \partial_x \partial_z$ - mixed partial

Fact: The order of diff is immaterial for polynomials.

If I want to approx f well by poly. it's mixed partials better not depend on order of diff.

Thm Suppose $D \subseteq \mathbb{R}^2$, open set $\vec{a} = (a,b) \in D$

$f : D \to \mathbb{R}$, & $\partial_x f$, $\partial_y f$

$\partial_y \partial_x f$ & $\partial_x \partial_y f$ are all cont. at $\vec{a}$.

Then $\partial_y \partial_x f(\vec{a}) = \partial_x \partial_y f(\vec{a})$.

Better Thm

Thm Suppose $D \subseteq \mathbb{R}^2$ is open, $f: D \to \mathbb{R}$, $\vec{a} \in D$ & $\partial_x f$, $\partial_y f$ & $\partial_y \partial_x f$ exist on D & are cont. at $\vec{a}$. Then $\partial_x \partial_y f$ exists at $\vec{a}$ & is $= \partial_y \partial_x f(\vec{a})$.

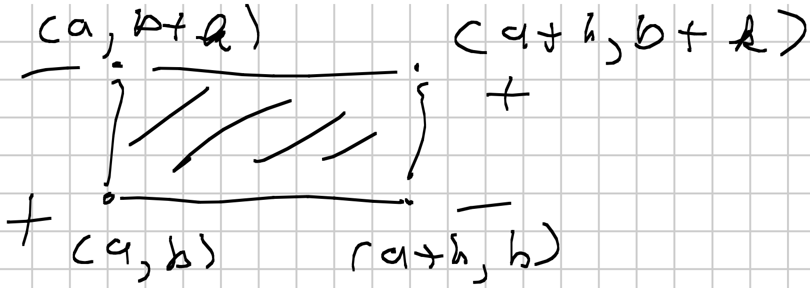
proof $\partial_y \partial_x f$ is cont. at $\vec{a} = (a, b)$. For $B_{2R}(\vec{a}) \subseteq D$.

$$\sup_{\vec{x} \in B_{2R}(\vec{a})} (\partial_y \partial_x f(\vec{x})) - \inf_{\vec{x} \in B_{2R}(\vec{a})} (\partial_y \partial_x f(\vec{x}))$$

$$= \varepsilon(R).$$

Cont. of $\partial_y \partial_x f$ at $\vec{a}$ implies

$$\lim_{R \to 0} \varepsilon(R) = 0$$



$$\star \quad \frac{f(a,b) - f(a+h,b) + f(a+h,b+k) - f(a,b+k)}{hk}$$

$$= \frac{\left(f(a+h,b+k) - f(a+h,b)\right) - \left(f(a,b+k) - f(a,b)\right)}{hk}$$

$$g(x) = f(x, b+k) - f(x, b)$$

$$\star = \boxed{\frac{g(a+h) - g(a)}{h}} \frac{1}{k}$$

g is diff on $[a, a+h]$

By MVT

$$\star = g'(c)\,\frac{1}{k} \qquad c \text{ - bet. } a \,\&\, a+h$$

$$= \frac{\partial_x f(c, b+k) - \partial_x (f(c,b))}{k}$$

$$\ell(y) = \partial_x f(c, y)$$

$$A = \frac{\ell(b+k) - \ell(b)}{k}$$

ℓ is diff on $[b, b+k]$

+ by MVT

$$A = \ell'(d)$$

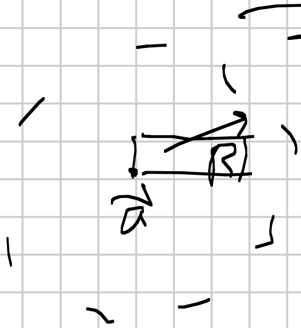
$$= \partial_y \partial_x f(c, d)$$

(c,d) is in rect

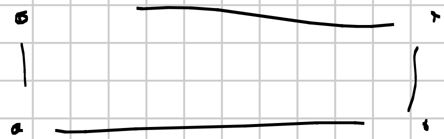
$$\boxed{\partial_y \partial_x f(c,d)}$$

So

$$\left|A - \partial_y \partial_x f(\vec{a})\right| \leq \varepsilon\left(\max(|h|, |k|)\right)$$



$$\star = \frac{f(a,b) - f(a+h,b) + f(a+h,b+k) - f(a,b+k)}{kh}$$



$$= \Big(\big(f(a+h,b+k) - f(a,b+k)\big) - \big(f(a+h,b) - f(a,b)\big) \Big) \Big/ kh$$

$$\left| \star - \partial_y \partial_x f(\vec{a}) \right| \le \varepsilon(\max(|h|,|k|))$$

Take limit as $\underline{k \to 0}$

Set
$g(y) = f(a+h,y) - f(a,y)$,

& lim. is diff g
w.r.t. y at b.

$$\left| \frac{\partial_y f(a+h,b) - \partial_y f(a,b)}{h} - \partial_y \partial_x f(\vec{a}) \right| \le \varepsilon(|h|)$$

as $h \to 0$

$$\varepsilon(|h|) \to 0$$

So $$\lim_{h\to 0} \frac{\partial_y f(a+h, b) - \partial_y f(a,b)}{h}$$

$$= \partial_y \partial_x f(\vec{a})$$

$$\partial_x \partial_y f(\vec{a}) = \partial_y \partial_x f(\vec{a})$$