Nov 6

Note Title 11/6/2009

Smooth sfces in $\mathbb{R}^3$

1) Near any pt of set one var. can be written as a C^1 fctn of the other 2.

2) level set of a C^1 fctn whose grad is not 0.

3) Parametrized sfc. open set $S \subseteq \mathbb{R}^2$

$g: S \to \mathbb{R}^3$ –

Dg – rank 2

Smoothness – should give us a tangent plane.

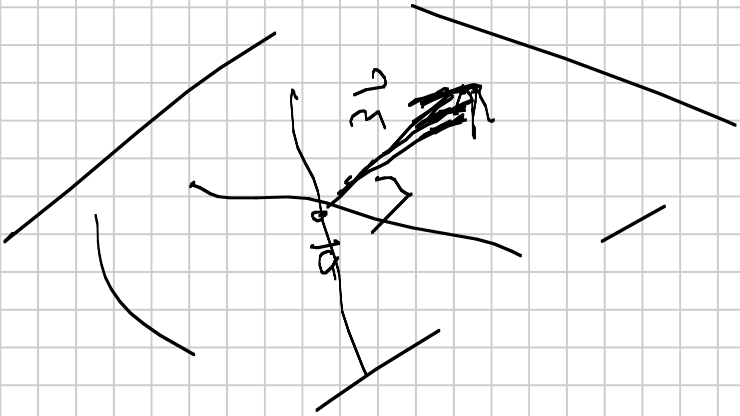
In $\mathbb{R}^3$ the eqn. of a plane looks like

$$\vec{n}\cdot(\vec{x}-\vec{a}) = 0$$

$\vec{a}$ - pt on plane

$\vec{n}$ - normal to plane

$\vec{n} \neq \vec{0}$.



2) $\underline{n = \nabla F}$

3) g - param. sfc.

$(x_0, y_0) \in S$

$\vec{g}(x_0, y_0) = \vec{a}$

Fix y_0

$\vec{g}(x_0 + h, y_0) =$

curve on sfc.

so $(\partial_x g_1(x_0,y_0), \partial_x g_2(x_0,y_0),$

$\partial_x g_3(x_0,y_0))$

is $\parallel$ sfc.

similarly

$(\partial_y g_1, \partial_y g_2, \partial_y g_3)$

is also $\parallel$ sfc.

These are the columns of

Dg.

So

$$D_x g \times D_y g = n$$

is a normal

& $\neq 0$.

Apply this to case 1.

$g(x,y) = (x, y, f(x,y))$

is the param.

if $z = f(x, y)$ is a C^1 fctn of $\underline{x + y}$.

$$Dg = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \partial_x f & \partial_y f \end{bmatrix}$$

$$n = D_x g \times D_y g$$

$$\begin{bmatrix} i & j & k \\ 1 & 0 & \partial_x f \\ 0 & 1 & \partial_y f \end{bmatrix} = i \cdot (-\partial_x f) - j(\partial_y f) + k$$

$$= (-\partial_x f, -\partial_y f, 1) = \vec{n}$$

eqn. of tan plane

$$\vec{n}(\vec{x} - \vec{a}) = 0$$

$$\vec{a} = (x_0, y_0, f(x_0, y_0))$$

$$\vec{x} = (x, y, z)$$

$$- \partial_x f(x_0, y_0)(x - x_0)$$
$$- \partial_y f(x_0, y_0)(y - y_0)$$
$$+ (z - f(x_0, y_0)) = 0$$

$$z = f(x_0, y_0) + \partial_x f(x_0, y_0)(x - x_0)$$
$$+ \partial_y f(x_0, y_0)(y - y_0).$$

1st order Taylor approx. to f.

Curves in $\mathbb{R}^3$.

1) Near every pt. 2 of var. can be written as C^1 fctns of third.

2) Sol. set of a pair of fctns

$$\vec{F} = (F_1, F_2) = 0.$$

+ $D\vec{F}$ - (2 x 3) matrix

has rank 2.

3) Param. curve

$$g(t) = (g_1(t), g_2(t), g_3(t))$$

$g - C^1$

$Dg \neq 0.$

Defn is ① + connected

A k dim set in $\mathbb{R}^n$, $n \geq k$, sat. ① + connected is called a "k-dim C^1-manifold-embedded in $\mathbb{R}^n$."

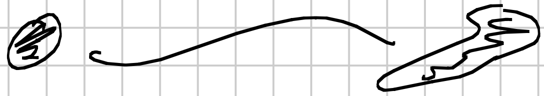
Inverse Fct. Theorem.

$S \subseteq \mathbb{R}^n$ - open

$F: S \to \mathbb{R}^n$

When is F invertible?

or at least locally near a pt?



Thm Suppose $S \subseteq \mathbb{R}^n$ - open

$F: S \to \mathbb{R}^n$

is C^1 -

$\vec{x}_0 \in S$

& $DF(\vec{x}_0)$ - invertible

then $\exists$ nghds

U of $\vec{x}_0$ & V of $\vec{y}_0 = F(\vec{x}_0)$

& For every $\vec{y} \in V$

$\exists$ a unique $\vec{x} \in U$

with $F(\vec{x}) = \vec{y}$

Letting $\vec{x} = G(\vec{y})$

G is C^1

& $DG(\underline{\tilde{y}_0}) = (DF(x_0))^{-1}$

pf Set

$$H(\tilde{x}, \tilde{y}) = \vec{y} - \underline{F}(\vec{x}).$$

0-set of H is the set where $\vec{y} = F(\vec{x})$.

$$H : \underset{\vec{x}\quad\quad\vec{y}}{\mathbb{R}^{2n} \to \mathbb{R}^n}$$

$$DH = \left[\; B \rightarrow \underbrace{-DF(\vec{x}_0)}_{\substack{\text{col are}\\ \text{lin ind.}}} \;\middle|\; \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \leftarrow A \;\right]$$

& IFT says there are $r_0, r_1 > 0$ & $\forall y \in B_{r_0}(y_0) = V$

$\exists$ a unique $x \in B_{r_1}(x_0) = U$

with $H(x, y) = 0$

i.e. $F(x) = y$.

F letting $\vec{x} = G(\vec{y})$

$$F(G(\vec{y})) = \vec{y}.$$

+ $DG(\vec{y}_0) = -B^{-1}A$

$$= (DF(x_1))^{-1}$$