Nov 11

$f : [a, b]$, $a < b$

f - bounded.

Upper sums

$$\overline{S}(f, P) = \sum_{i=1}^{n} M_i \, \ell(I_i)$$

$$\underline{S}(f, P) = \sum_{i=1}^{n} m_i \, \ell(I_i)$$

$\mathcal{U} = \{$upper sums$\}$

$\mathcal{L} = \{$lower sums$\}$

All elts of $\mathcal{U}$ are $\geq$ all elts of $\mathcal{L}$.

$$\overline{I}_a^b(f) = \inf(\mathcal{U})$$

$$\underline{I}_a^b(f) = \sup(\mathcal{L})$$

If $\overline{I}_a^b(f) = \underline{I}_a^b(f)$

we say f is Riemann integrable on $[a,b]$ &

$$\int_a^b f(x)\,dx = \overline{I}_a^b(f) = \underline{I}_a^b(f).$$

<u>Thm</u> $f:[a,b]\to\mathbb{R}$, bounded. Then f is R. integrable iff for all $\varepsilon>0$ $\exists$ partition P with

$$\overline{S}(f,P) - \underline{S}(f,P) < \varepsilon.$$

<u>pf</u>. $\Leftarrow$ Let $\varepsilon>0$, & P s.t. $\overline{S}(f,P) - \underline{S}(f,P) < \varepsilon$.

$$\overline{S}(f,P) \geq \overline{I}_a^b(f)$$

$$\underline{S}(f,P) \leq \underline{I}_a^b(f)$$

$$0 \le \overline{I}_a^{\,b}(f) - \underline{I}_a^{\,b}(f) \le \varepsilon .$$

$$\Rightarrow \overline{I}_a^{\,b} = \underline{I}_a^{\,b} \checkmark .$$

$\Rightarrow$ Suppose

f is int.

Select P_1

$$0 \le \overline{S}(f, P_1) - \overline{I}_a^{\,b}(f) < \varepsilon/2 .$$

& P_2

$$0 \le \underline{I}_a^{\,b}(f) - \underline{S}(f, P_2) < \varepsilon/2$$

$$0 \le \overline{S}(f, P_1) - \underline{S}(f, P_2) < \varepsilon$$

$$P = P_1 \cup P_2$$

&

$$\underline{S}(f, P_2) \le \underline{S}(f, P) \le \overline{S}(f, P) \le \overline{S}(f, P_1)$$

& $\overline{S}(f, P) - \underline{S}(f, P) < \varepsilon .$

Defn Suppose
$P = \{x_0 < x_1 < \dots < x_n\}$
is a partition of $[a,b]$.
Then "mesh" of P
is $\max((x_i - x_{i-1}))$ =
max of lengths
of intervals.

Thm $f : [a,b] \to \mathbb{R}$ -
bounded.
$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. if
$\text{mesh}(P) < \delta$ then

$$\overline{S}(f,P) - \overline{I}_a^b(f) < \varepsilon.$$

& $$\underline{I}_a^b(f) - \underline{S}(f,P) < \varepsilon.$$

proof Assume $|f(x)| \le B$
Select Q so that

$$\overline{S}(f,Q) - \overline{I}_a^b(f) < \varepsilon/2$$

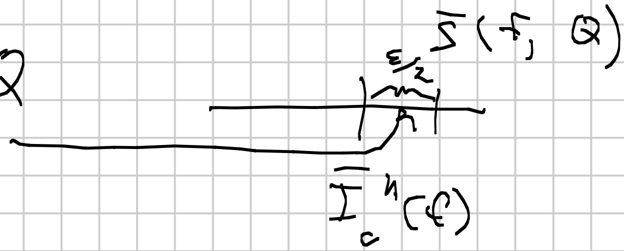
$Q = \{y_0, y_1, \cdots, y_m\}$.

$$\delta = \frac{\varepsilon}{4mB}$$

Suppose mesh$(P) < \delta$.

First look at

$P \cup Q$



$$\overline{S}(f, P \cup Q) - \overline{I}_a^b(f) < \varepsilon/2$$

I know

$\overline{S}(f, P)$ can be

bigger than $\overline{S}(f, P \cup R)$

How much?

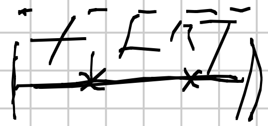
Removing at most

$(m-1)$ pts.

This effects at

most $(m-1)$ of

intervals of P.



inf.

Change is bdd by

$$B\,\underline{\ell(I_i)} - (-B)\,\ell(I_i)$$

$$2\underline{B}\,\ell(I_i)$$

Total change possible

$$(\#\text{int})\cdot 2B\,\delta$$

$$= 2B\,\delta\,(n-1)$$

$$= \frac{\cancel{2}(n-1)\cancel{B}\,\varepsilon}{2\cdot\cancel{4}\,n\,\cancel{B}} < \frac{\varepsilon}{2}$$

$$0 \le \overline{S}(f,P) - \overline{S}(f,P\vee Q) < \varepsilon/2$$

$$\& \quad \overline{S}(f,P) - \overline{I}_a^b(f) < \varepsilon.$$

Defn $P_n = \{a, a+\frac{b-a}{n}, a+2\frac{b-a}{n}, \ldots, b\}$

Cor

$$\overline{I}_a^b(f) = \lim_{n\to\infty} \overline{S}(f, P_n)$$

$$\underline{I}_a^b(f) = \lim_{n\to\infty} \underline{S}(f, P_n)$$

Thm Suppose $a < b < c$

f is int. on $[a,b]$

& on $[b,c]$

Then f is int.

on $[a,c]$ &

$$\int_a^c f = \int_a^b f + \int_b^c f$$

Pf Let Q_n partition

$[a,b]$, $\text{mesh}(Q_n) \searrow 0$

& $\mathcal{H}_n$ partition

$[b,c]$, $\operatorname{mesh}(H_n) \searrow 0$

Let L_n ~~[illegible]~~ $= G_n \cup H_n$

a partition of $[a,c]$.

$\operatorname{mesh}(L_n) \searrow 0$

& $\overline{S}(f, L_n) = \overline{S}(f, G_n) + \overline{S}(f, H_n)$

& $\overline{I}_a^c(f) = \overline{I}_a^b(f) + \overline{I}_b^c(f)$

&

$\underline{I}_a^c(f) = \underline{I}_a^b(f) + \underline{I}_b^c(f)$

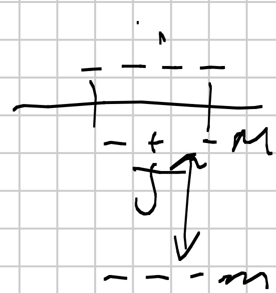
<u>Thm</u> If $f : [a,b]$ is bounded & R. int. then $|f|$ is also int.

& $\int_a^b |f| \geq \left| \int_a^b f \right|$

Pf. Notice

$$\overline{S}(g,P) - \underline{S}(g,P)$$

$$= \sum_{i=1}^{m} (M_i - m_i)\, \ell(I_i)$$



M - sup

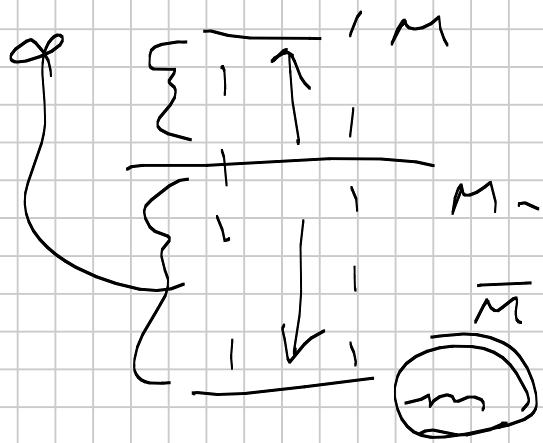
m - inf

for f

$\overline{M}$ - sup

$\overline{m}$ - inf

for $|f|$



$M - m$

$\overline{M} - \overline{m}$

$$\overline{M} - \overline{m} \le M - m$$