Dec 2

Note Title 12/2/2009

Content 0
vs
Measure 0

Note: If C is cpct
& of 0-measure
then it is of 0-content.

Int. in 2-dim.

Direct product of sets

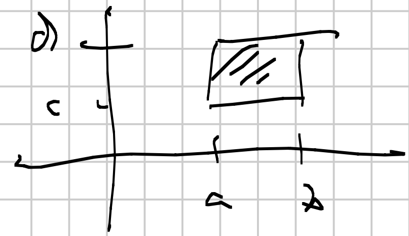
A, B sets.

$A \times B = \{(a,b) : a \in A, b \in B\}$

I, J ~ intervals in $\mathbb{R}$

$I \times J$ = Rectangle

$[a,b] \times [c,d]$

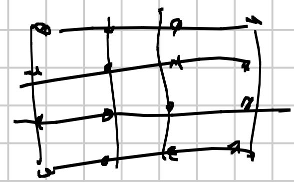


Partitions —

P_1 - part. of I

P_2 - part of J

$P_1 \times P_2$



$$P = \{ I_i \times J_j : I_i - \text{int of } P_1, \; J_j - \text{int of } P_2 \}$$

$$= \{ R_{i,j} \}$$

Def. upper & lower sums

$$\overline{S}(f, P) = \sum_{i,j} M_{i,j} \, \text{Area}(R_{i,j})$$

n

$$\underline{S}(f,P) = \sum_{i,j} m_{i,j} \operatorname{Area}(B_{i,j})$$

$\mathcal{U}(f)$ - $\{$all upper sums$\}$

$\mathcal{L}(f)$ - $\{$all lower sums$\}$

all $\mathcal{U}(f)$
above all
$\mathcal{L}(f)$

$$\overline{I}_B(f) = \inf(\mathcal{U}(f))$$

$B = I \times J$

$$\underline{I}_B(f) = \sup(\mathcal{L}(f))$$

$$\overline{I}_B(f) \geq \underline{I}_B(f).$$

If $\overline{I}_B(f) = \underline{I}_B(f)$

Then we say f is R. int on B.

$$\int_B f \, dA \qquad \left(\int_c^d \int_a^b f(x,y)\, dx\, dy \right)$$

1) Mesh(P)

$= \max(\text{Mesh}P_1, \text{Mesh}P_2)$

<u>Thm</u> If f is bded.

& $\varepsilon > 0$ then $\exists \delta > 0$

& if Mesh$(P) < \delta$.

then

$$0 \le \overline{S}(f,P) - \overline{I}_B(f) < \varepsilon$$

$$0 \le \underline{I}_B(f) - \underline{S}(f,P) < \varepsilon.$$

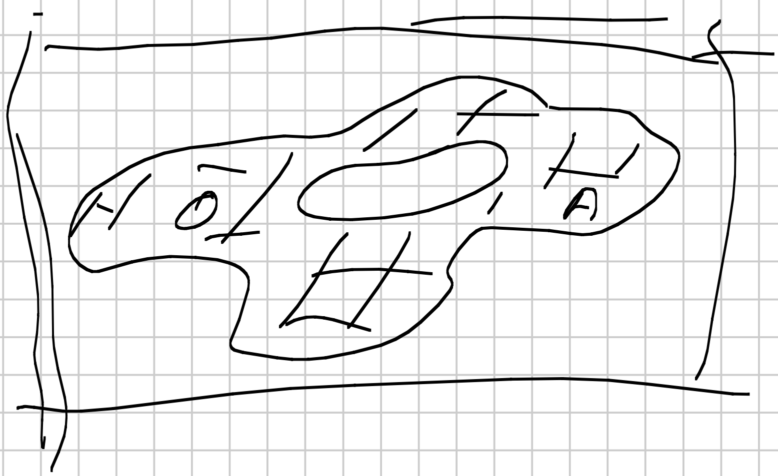
<u>Thm</u> If f is bded

+ the disc. of f

have content 0

then f is <u>in</u>tegrable.

(iff the set of disc has measure 0).



Put A in a box!

Characteristic or indicator function

$$\chi_A, \ I_A, \ \mathbb{1}_A$$

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

$$\int_A f \, dA = \int_B \chi_A f \, dA$$

χ_A - almost never continuous

Lemma $A \subseteq \mathbb{R}^2$,

$\text{disc.}(\chi_A) = \partial A$.

pf. $x \in \partial A$, $\exists x_i \in A$, $x_i \to x$

& $y_i \in A^c$, $y_i \to x$

but $\lim_{i \to \infty} \chi_A(x_i) = 1$

$\lim_{i \to \infty} \chi_A(y_i) = 0$

$x \in A^{int}$, $\exists r > 0$,

$B_r(x) \subseteq A$,

once $\|\bar{x} - \bar{y}\| < r$

$\chi_A(x) = \chi_A(y)$

$x \in (A^c)^{int}$

$\Rightarrow \exists r > 0$

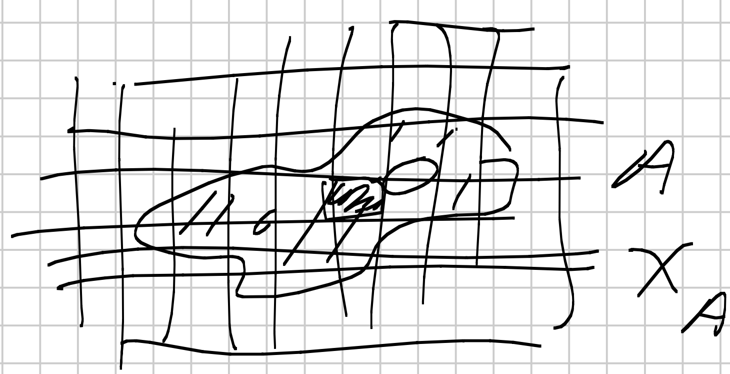
$B_r(x) \subseteq A^c.$

Thm Suppose $A \subseteq \mathbb{R}^2$ is bounded. Then χ_A is integrable iff ∂A has content 0.

Use ∂A - cpct.

Area

We say A is "measurable" if χ_A is integrable ie ∂A has 0-content.



$$\overline{S}(P, \chi_A) = \left(\text{sum of areas of all rect. that intersect } A \right)$$

$$\underline{S}(P, \chi_A) = \left(\text{sum of areas of all rect cont. in } A \right)$$

$\overline{I}_B(\chi_A)$ - outer area of $\underline{A}$,

$\underline{I}_B(\chi_A)$ - inner area of $\underline{A}$