Note Title 8/31/2009

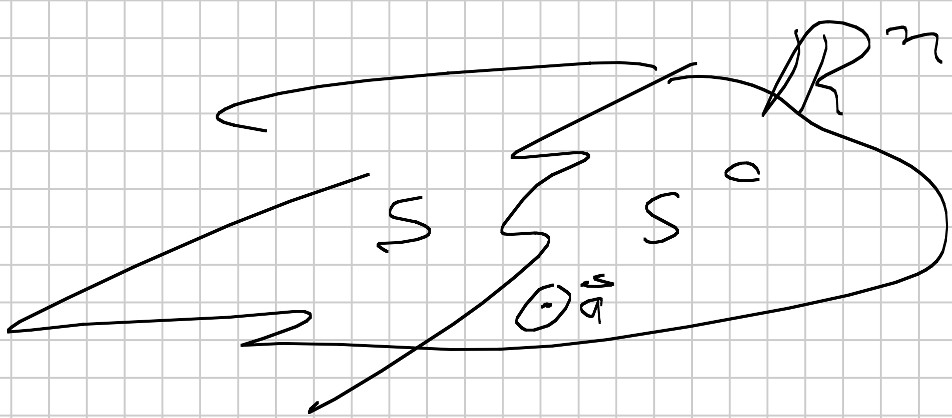
Aug 31

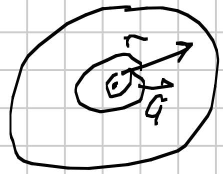
Open sets

Closed sets.

def $\vec{a} \in S^{int}$
S^{o}
if $\exists r > 0$ & $B_r(\vec{a}) \subseteq S$.

$\vec{a} \in \partial(S)$ if $\forall r > 0$,
$B_r(\vec{a}) \cap S \neq \emptyset$ &
$B_r(\vec{a}) \cap S^c \neq \emptyset$.





A set is open if

$$\partial(S) = \emptyset$$

A set is closed if

$$\partial(S) \subseteq S.$$

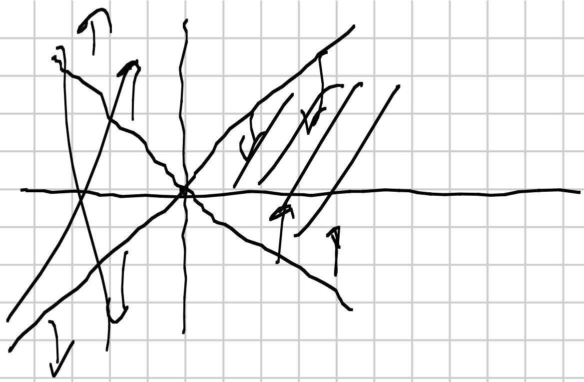
HW p. 12, #1, a. b. c. g.

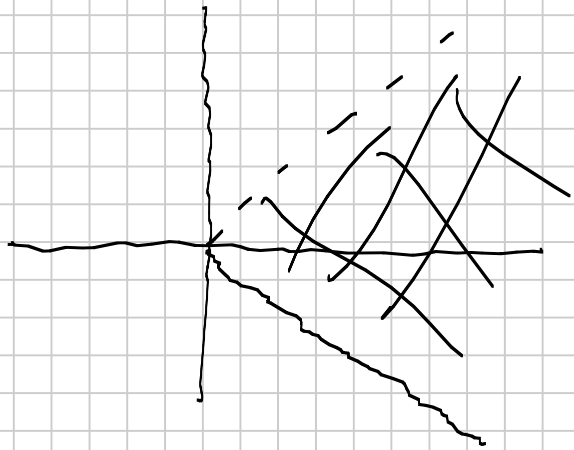
Ex. $S = \{(x, y) \in \mathbb{R}^2 : -x < y < x\}$

1) Sketch S

2) is S open, closed, neither

3) What is S^{int}, $\overline{S}$, $\partial(S)$.





$$\partial(S) = \{(x,y) : y = x \text{ or } y = -x,\ x \geq 0\}.$$

$$S^{int} = \{(x,y) : -x < y < x,\ x \geq 0\}$$

is S open? NO

S closed? NO

Neither

$\overline{S}$ - closure $= S \cup \partial S$.

$= \{(x,y) : -x \le y \le x,$
$x \ge 0\}$.

<u>Lemma</u> Suppose

$S \subseteq \mathbb{R}^n$

$\vec{a} \in S$, $r > 0$ &

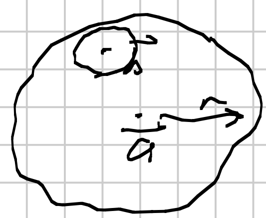
$B_r(\vec{a}) \subseteq S$.

Then $B_r(\vec{a}) \subseteq S^{int}$.

<u>Pf</u> Need to show if

$\vec{b} \in B_r(\vec{a})$ then $\exists\, r' > 0$

& $B_{r'}(\vec{b}) \subseteq S$.



$\vec{b} \in B_r(\vec{a})$ so

$\|\vec{a} - \vec{b}\| < r$

$r' = r - \|\vec{a} - \vec{b}\| > 0$.

Claim $B_{r'}(\vec{b}) \subseteq B_r(\vec{a})$.

Why? suppose

$\vec{c} \in B_{r'}(\vec{b})$ //

$\Rightarrow \quad \|\vec{b}-\vec{c}\| < r' = r - \|\vec{a}-\vec{b}\|$

$$\underbrace{\|\vec{a}-\vec{b}\| + \|\vec{b}-\vec{c}\|}_{\geq\!\!/} < r.$$

$\|\vec{a}-\vec{c}\| < r$

$\vec{c} \in B_r(\vec{a})$ //

$B_{r'}(\vec{b}) \subseteq \underline{B_r}(\vec{a}).$

So $\quad B_{r'}(\vec{b}) \subseteq \underline{\underline{B_r}}(\vec{a}) \subseteq S.$

$\Rightarrow \quad \underline{\underline{\vec{b} \in S^{int}}}$

$$\underline{\left| \; B_r(\vec{a}) \subseteq S^{int} \; \right|}$$

Thm

1) $(S^{int})^{int} = S^{int}$

2) $\forall S$, S^{int} is open.

3) $\vec{a} \in \mathbb{R}^n$, $r > 0$,

$B_r(\vec{a})$ is open.

4) Let

$$\mathcal{B}(S) = \{ B_r(\vec{a}) : B_r(\vec{a}) \subseteq S \}$$

$$S^{int} = \bigcup_{B \in \mathcal{B}(S)} B.$$

pf. 1) if $A \subseteq B$ then

$A^{int} \subseteq B^{int}$.

$S^{int} \subseteq S$

$\Rightarrow (S^{int})^{int} \subseteq S^{int}$

$\vec{a} \in S^{int} \Rightarrow \exists r > 0 \;\; B_r(\vec{a}) \subseteq S$

Lemma $\Rightarrow B_r(\vec{a}) \subseteq S^{int}$

$$\Rightarrow \vec{a} \in (S^{int})^{int}$$

$$\underline{\underline{\overbrace{S^{int} \subseteq (S^{int})^{int}}}}$$

2) A - open iff $A^{int} = A$

$$(S^{int})^{int} = S^{int} - S^{int} \text{ open}$$

3) If $\vec{b} \in B_r(\vec{a}) \Rightarrow \vec{b} \in B_r(\vec{a})^{int}$

$$\underline{B_r(\vec{a})^{int} = B_r(\vec{a})}$$

4) $\mathcal{B}(S) = \{ B_r(\vec{a}) : B_r(\vec{a}) \subseteq S \}$

$$B_r(S) \in \mathcal{B}(S)$$

by lemma

$$B_r(S) \subseteq S^{int}$$

$$\Rightarrow \bigcup_{B \in \mathcal{B}(S)} B \subseteq S^{int}$$

$\vec{a} \in S^{int} \Rightarrow \exists r > 0,\ B_r(\vec{a}) \in \mathcal{B}(S)$

$$\Rightarrow \vec{a} \in \bigcup_{B \in \mathcal{B}(S)} B$$

$$S^{int} \subseteq \bigcup_{B \in \mathcal{B}(S)} B.$$

$$S^{int} = \bigcup_{B \in \mathcal{B}(S)} B.$$

Thm If $\mathcal{A}$ is any collection of open sets

& $S = \bigcup_{A \in \mathcal{A}} A$

Then S is open.

pf $\vec{a} \in S \Rightarrow \exists A \in \mathcal{A}$, $\vec{a} \in A$. A is open so $\exists r > 0$, $B_r(\vec{a}) \subseteq A$

$$\Rightarrow B_r(\vec{a}) \subseteq S$$

$$\Rightarrow S^{int} = S.$$

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