Aug 28

Note Title 8/28/2009

Decimal expansions

Floor function

$\lfloor x \rfloor$ = largest integer $\le x$.

Fractional part

$\{x\} = x \bmod 1 = x - \lfloor x \rfloor$

$\in [0,1)$.

Two inf. series

$$\underset{\parallel}{\sum_{i=1}^{\infty} a_i}, \quad \underset{\parallel}{\sum_{i=1}^{\infty} b_i}$$

$$A \qquad\qquad B$$

Suppose $a_i \geq b_i$.

$A \geq B$

Suppose $a_i \geq b_i$

$\&$ $A = B$.

$\Rightarrow$ all $\underline{\underline{a_i = b_i}}$

nine symbols

$\underline{0, 1, 2 \cdots, 9.}$

$a_i \quad i = 1, 2, \cdots$

$a_i \in \{0, 1, \cdots, 9\}$.

$$.a_1 a_2 a_3 \cdots = \sum_{i=1}^{\infty} \frac{a_i}{10^i}.$$

$$0 \leq \frac{a_i}{10^i} \leq \frac{9}{10^i}.$$

$$\sum_{i=1}^{\infty} \frac{0}{10^i} \leq .a_1 a_2 \cdots \leq \sum_{i=1}^{\infty} \frac{9}{10^i}.$$

$$\sum_{i=1}^{\infty} \frac{0}{10^i} = 0$$

$$\sum_{i=1}^{\infty} \frac{9}{10^i} = \frac{9/10}{1 - 1/10} = 1$$

$\in [0, 1]$.

Thm Suppose

$$.a_1 a_2 \cdots = .b_1 b_2 \cdots$$

Then either

1) $a_i = b_i$ for all i

or

2) There is a k_0

$\ni$

$$.a_1 a_2 \ldots = .a_1 a_2 \cdots \underline{a_{k_0}}\, 000\overline{0}$$

$a_{k_0} \neq 0$ &

$$.b_1 b_2 \ldots = .a_1 a_2 \cdots \underline{(a_{k_0} - 1)}\, 99\ldots\overline{9}$$

pf. Suppose $\exists k_0$ $a_{k_0} \neq b_{k_0}$ – let k_0 be the first such & assume $a_{k_0} > b_{k_0}$.

$$x = .a_1 a_2 \cdots \geq .a_1 a_2 \cdots a_{k_0} 000\overline{0}$$

$$y = .b_1 b_2 \cdots = .a_1 a_2 \cdots a_{k_0 - 1} \underline{b_{k_0}}\, b_{k_0+1} \cdots$$

$$\leq .a_1 a_2 \cdots \underbrace{a_{k_0-1} b_{k_0}}\, 9\, 9 \cdots \bar{9}$$

$$= a_1 a_2 \cdots a_{k_0-1} (b_{k_0}+1)\, 0\, 0 \cdots \bar{0}$$

$$x - y \geq .a_1 a_2 \cdots a_{k_0} 0 \bar{0} - .a_1 a_2 \cdots (b_{k_0}+1) \bar{0}$$

$$= \frac{a_{k_0} - (b_{k_0}+1)}{10^{k_0}} \quad, = 0$$

$$\underline{a_{k_0} = b_{k_0} + 1}$$

$$\Rightarrow \quad .a_1 a_2 \cdots a_{k_0} 0\, 0 \cdots \bar{0} \quad \leq x$$

$$= .a_1 a_2 \cdots (a_{k_0} - 1)\, 9\, 9 \cdots \bar{9} \geq y$$

$$.a_1 a_2 \cdots = .a_1 a_2 \cdots a_{k_0} \bar{0}$$

$$.b_1 b_2 \cdots = .a_1 a_2 \cdots (a_{k_0} - 1) \bar{9}.$$

$$.a_1 a_2 \cdots \longrightarrow x = \sum_{i=1}^{\infty} \frac{a_i}{10^i}.$$

$$x \quad , a_1 = \lfloor 10x \rfloor$$

$$a_2 = \lfloor 10(x - a_1) \rfloor$$

$$a_k = \lfloor 10^k x \bmod 10 \rfloor$$

When is the dec. exp rational?

<u>Thm</u> x is rat'l iff its dec. exp. is eventually periodic.

$$x = .\underbrace{a_1 a_2 \cdots a_{k_0}}[\underbrace{b_1 \cdots b_p}]_\infty$$

$$10^{k_0} x = n + .\overline{b_1 b_2 \cdots b_p}$$

$$10^{k_0+p} x = m + .\overline{b_1 b_2 \cdots b_p}$$

$$(10^{k_0+p} - 10^{k_0})x = m - n$$

$$x = \frac{m-n}{10^{k_0+p} - 10^{k_0}} \in \mathbb{Q}.$$

x - rat'l.

$$x = \frac{p}{q}$$

$$a_k = \lfloor 10^k x \bmod 10 \rfloor$$

$$10^k \left(\frac{p}{q}\right) \bmod 10$$

rat'l, $0 \leq \; \leq 10$

denom $\leq q$.

only fin many.

$$\lfloor 10^k x \bmod 10 \rfloor$$

Believe

Lemma: If

$$a_1 = b_1, \quad a_2 = b_2, \quad \cdots, \quad a_{k_0} = b_{k_0}$$

then

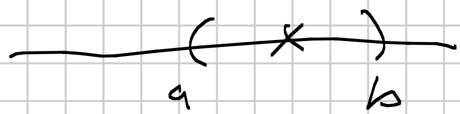
$$\| .a_1 a_2 \cdots, \; .b_1 b_2 \cdots \| \leq \frac{1}{10^{k_0}}$$

both $.a_1 a_2 \cdots$ + $.b_1 b_2 \cdots$

are in

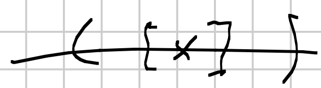
$$\left[\,.a_1 a_2 \cdots a_{k_0}\overline{0},\ .a_1 a_2 \cdots a_{k_0}\overline{9}\,\right]$$

This int. has length $\frac{1}{10^{k_0}}$.



$c = \frac{a+b}{2}$, choose k_0 so that

$$\left[c - \frac{1}{2\cdot 10^{k_0}},\ c + \frac{1}{2\cdot 10^{k_0}}\right] \subseteq (a,b)$$



write

$$c = .c_1 c_2 \cdots c_{k_0} \cdots$$

$q = .c_1 c_2 \cdots c_{k_0}\overline{0}$ is rat'l

$$|c - q| \le \frac{1}{10^{k_0}}.$$

How about an irrat'l?

$$r = .c_1 c_2 \ldots c_{k_0} 0100100010000100000 1 \ldots$$