Poincaré Duality Angles on Riemannian Manifolds with Boundary

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Let $M^n$ be a compact Riemannian manifold with non-empty boundary $\partial M$. 
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Figure 1. The concrete realizations of the absolute and relative cohomology groups $H^p(M;\mathbb{R})$ and $H^p(M,\partial M;\mathbb{R})$. 

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de Rham’s Theorem

Suppose $M^n$ is a compact, oriented, smooth manifold. Then

$$H^p(M; \mathbb{R}) \cong C^p(M)/\mathcal{E}^p(M),$$

where $C^p(M)$ is the space of closed $p$-forms on $M$ and $\mathcal{E}^p(M)$ is the space of exact $p$-forms.
If $M$ is Riemannian, the metric induces an $L^2$ inner product on $\Omega^p(M)$:

$$\langle \omega, \eta \rangle := \int_M \omega \wedge \star \eta.$$
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When $M$ is closed, the orthogonal complement of $\mathcal{E}^p(M)$ inside $\mathcal{C}^p(M)$ is

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Kodaira called this the space of harmonic $p$-fields on $M$. 

Hodge’s Theorem

If $M^n$ is a closed, oriented, smooth Riemannian manifold,

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M).$$
Define $i : \partial M \hookrightarrow M$ to be the natural inclusion.
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The $L^2$-orthogonal complement of the exact forms inside the space of closed forms is now:

$$\mathcal{H}^p_N(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta \omega = 0, i^* \star \omega = 0 \}.$$
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Then

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p_N(M).$$
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\[ H^p(M, \partial M; \mathbb{R}) \cong \mathcal{H}^p_D(M). \]
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\[ \mathcal{H}^p_D(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^*\omega = 0 \}. \]
The concrete realizations of $H^p(M; \mathbb{R})$ and $H^p(M, \partial M; \mathbb{R})$ meet only at the origin:

$$\mathcal{H}_N^p(M) \cap \mathcal{H}_D^p(M) = \{0\}$$
Non-orthogonality

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...but they are not orthogonal!
Interior and boundary subspaces

Interior subspace of $\mathcal{H}_N^p(M)$:

$$\ker i^* \text{ where } i^* : H^p(M; \mathbb{R}) \to H^p(\partial M; \mathbb{R})$$
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$$\mathcal{E}_\partial \mathcal{H}_N^p(M) := \{ \omega \in \mathcal{H}_N^p(M) : i^* \omega = d\varphi, \varphi \in \Omega^{p-1}(\partial M) \}.$$
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Interior subspace of $\mathcal{H}_D^p(M)$:

$$\ast \mathcal{E}_{\partial} \mathcal{H}_N^{n-p}(M) = c \mathcal{E}_{\partial} \mathcal{H}_D^p(M)$$

$$= \{ \eta \in \mathcal{H}_D^p(M) : i^* \ast \eta = d\psi, \psi \in \Omega^{n-p-1}(\partial M) \}.$$
To prove that there is an element of $c_{EH}^p N(M)$ having arbitrary preassigned periods on $c_p^1 \hookrightarrow \ldots c_p^g$, it suffices to show that $(F_1 \hookrightarrow \ldots \hookrightarrow F_g) \mapsto (C_1 \hookrightarrow \ldots \hookrightarrow C_g)$ is an isomorphism.

Suppose some set of $F_i$-values gives all zero $C_i$-values, meaning that $i^* \eta$ is zero in the cohomology of $\partial M$. In other words, the form $i^* \eta$ is exact, meaning that $\eta \in E_{\partial H}^p N(M)$, the interior subspace of $H^p N(M)$. Since $E_{\partial H}^p N(M)$ is orthogonal to $c_{EH}^p N(M)$, this implies that $\eta = 0$, so $\tilde{\eta} = \pm \star \eta = 0$ and hence the periods $F_i$ of $\tilde{\eta}$ must have been zero.

Therefore, the map $(F_1 \hookrightarrow \ldots \hookrightarrow F_g) \mapsto (C_1 \hookrightarrow \ldots \hookrightarrow C_g)$ is an isomorphism, completing Step 1.

Step 2: Let $\omega \in H^p N(M)$ and let $C_1 \hookrightarrow \ldots \hookrightarrow C_g$ be the periods of $\omega$ on the above $p$-cycles $c_p^1 \hookrightarrow \ldots \hookrightarrow c_p^g$. Let $\alpha \in c_{EH}^p N(M)$ be the unique form guaranteed by Step 1 having the same periods on this homology basis.

Then $\beta = \omega - \alpha$ has zero periods on the $p$-cycles $c_p^1 \hookrightarrow \ldots \hookrightarrow c_p^g$; since $\beta$ is a closed form on $M$, it certainly has zero period on each $p$-cycle of $\partial M$ which bounds in $M$. Hence, $\beta$ has zero periods on all $p$-cycles of $\partial M$, meaning that $i^* \beta$ is exact, so $\beta \in E_{\partial H}^p N(M)$.

Therefore, $\omega = \alpha + \beta \in c_{EH}^p N(M) + E_{\partial H}^p N(M)$, so $H^p N(M)$ is indeed the sum of these two subspaces, as claimed in (2.1.8). This completes the proof of the theorem.

**Theorem 2.1.2** allows the details of Figure 1.1 to be filled in, as shown in Figure 2.1.

**Definition (DeTurck–Gluck)**

The *Poincaré duality angles* of the Riemannian manifold $M$ are the principal angles between the interior subspaces.
What do the Poincaré duality angles tell you?

Guess
If $M$ is “almost” closed, the Poincaré duality angles of $M$ should be small.
Consider $\mathbb{C}P^2$ with its usual Fubini-Study metric. Let $p \in \mathbb{C}P^2$. Then define

$$M_r := \mathbb{C}P^2 - B_r(p).$$
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$\partial M_r$ is a 3-sphere.
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\[ M_r \text{ has relative cohomology in dimensions 2 and 4.} \]

Therefore, \( M_r \) has a single Poincaré duality angle \( \theta_r \) between \( \mathcal{H}_N^2(M_r) \) and \( \mathcal{H}_D^2(M_r) \).
So the goal is to find closed and co-closed 2-forms on $M_r$ which satisfy Neumann and Dirichlet boundary conditions.
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Find closed and co-closed $SU(2)$-invariant forms on $M_r$ satisfying Neumann and Dirichlet boundary conditions.
The Poincaré duality angle for $M_r$

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Generalizes to $\mathbb{C}P^n - B_r(p)$. 
Consider

\[ N_r : = G_2 \mathbb{R}^n - \nu_r (G_1 \mathbb{R}^{n-1}) . \]
Consider

\[ N_r := G_2 \mathbb{R}^n - \nu_r(G_1 \mathbb{R}^{n-1}). \]

**Theorem**

- As \( r \to 0 \), all the Poincaré duality angles of \( N_r \) go to zero.
- As \( r \) approaches its maximum value of \( \pi/2 \), all the Poincaré duality angles of \( N_r \) go to \( \pi/2 \).
Conjecture

If $M^n$ is a closed Riemannian manifold and $N^k$ is a closed submanifold of codimension $\geq 2$, the Poincaré duality angles of $M - \nu_r(N)$

$\quad m - \nu_r(N)$

go to zero as $r \to 0$. 
What can you learn about the topology of $M$ from knowledge of $\partial M$?
Induce potentials on the boundary of a region and determine the conductivity inside the region by measuring the current flux through the boundary.
Electrical Impedance Tomography

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The Voltage-to-Current map

Suppose $f$ is a potential on the boundary of a region $M \subset \mathbb{R}^3$. 
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If $\gamma$ is the conductivity on $M$, the current flux through $\partial M$ is given by

$$(\gamma \nabla u) \cdot \nu = -\gamma \frac{\partial u}{\partial \nu}$$
The map $\Lambda_{\text{cl}} : C^\infty(\partial M) \to C^\infty(\partial M)$ defined by

$$f \mapsto \frac{\partial u}{\partial \nu}$$

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**Theorem (Lee-Uhlmann)**

*If $M^n$ is a compact, analytic Riemannian manifold with boundary, then $M$ is determined up to isometry by $\Lambda_{cl}$.***
Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

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$$\Lambda : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M)$$

**Theorem (Belishev–Sharafutdinov)**

The data $(\partial M, \Lambda)$ completely determines the cohomology groups of $M$. 
Define the *Hilbert transform* $T := d\Lambda^{-1}$. 

Theorem

If $\theta_1, \ldots, \theta_k$ are the Poincaré duality angles of $M$ in dimension $p$, then the quantities

$(-1)^{np + n + p} \cos^2 \theta_i$

are the non-zero eigenvalues of an appropriate restriction of $T^2$. 
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Belishev and Sharafutdinov posed the following question:

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**Theorem**

*The mixed cup product*

\[
\cup : H^p(M; \mathbb{R}) \times H^q(M, \partial M; \mathbb{R}) \rightarrow H^{p+q}(M, \partial M; \mathbb{R})
\]

is completely determined by the data \((\partial M, \Lambda)\) when the relative class is restricted to come from the boundary subspace.
• Poincaré duality angles for $G_4\mathbb{R}^8 - \nu_r(G_3\mathbb{R}^7)$? Other “Grassmann manifolds with boundary”?
Some questions

• Poincaré duality angles for $G_4 \mathbb{R}^8 - \nu_r(G_3 \mathbb{R}^7)$? Other “Grassmann manifolds with boundary”?

• What is the limiting behavior of the Poincaré duality angles as the manifold “closes up”?

• Can the full mixed cup product be recovered from $(\partial M, \Lambda)$? What about other cup products?

• Can the $L^2$ inner product on $H^p N(M)$ and $H^p D(M)$ be recovered from $(\partial M, \Lambda)$?
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- What is the limiting behavior of the Poincaré duality angles as the manifold “closes up”?
- Can the full mixed cup product be recovered from $(\partial M, \Lambda)$? What about other cup products?
- Can the $L^2$ inner product on $\mathcal{H}_N^p(M)$ and $\mathcal{H}_D^p(M)$ be recovered from $(\partial M, \Lambda)$?
Thanks!