Math 261 Exam 1 Review Formulas & Reminders

Dan Bates, Spring 2015

Here are a few formulas that might be handy for Exam 1. You *cannot* bring this to the exam, but hopefully it helps with studying....

WARNING: I do not guarantee that this is a comprehensive list! Also, please note that there are various alternative formulations for some of these formulas – I am just picking those that I like the best. Finally, there could be typos – beware!

- (Vector between two points) Given points (p_1, p_2, p_3) and (q_1, q_2, q_3) in $\mathbb{R}^3$, the vector between them is just $Q - P = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$.
- (Length of a vector) $|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.
- (Make a vector unit length) Just divide the vector by its length: $\frac{\mathbf{v}}{|\mathbf{v}|}$.
- (Dot product) $\mathbf{v} \cdot \mathbf{w} = \langle v_1, v_2, v_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$ (a number!) Recall that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.
- (Another dot product formula) $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- (Projection) The projection of vector $\mathbf{u}$ onto vector $\mathbf{v}$ is $\mathtt{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}$.
  The scalar component of the projection is $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$ (since $\frac{\mathbf{v}}{|\mathbf{v}|}$ is the unit-length direction).
- (Cross product) $\mathbf{v} \times \mathbf{w} = \langle v_1, v_2, v_3 \rangle \times \langle w_1, w_2, w_3 \rangle = \langle v_2 w_3 - v_3 w_2, -(v_1 w_3 - v_3 w_1), v_1 w_2 - v_2 w_1 \rangle$ (a vector!). Don't forget to negate that middle coordinate! (It might be easier to remember the 3 Xs way I taught you to compute the cross product.) Recall that two nonzero vectors are parallel if and only if their cross product is the zero vector.
- (Area of a triangle) The area of a triangle with edges $\mathbf{v}$ and $\mathbf{w}$ is $\frac{|\mathbf{v} \times \mathbf{w}|}{2}$. The volume of a parallelepiped with edges $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$.
- (Another cross product formula) $\mathbf{v} \times \mathbf{w} = (|\mathbf{v}||\mathbf{w}| \sin\theta)\, \mathbf{n}$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$ and n is a unit vector in the normal direction (orthogonal to $\mathbf{v}$ and $\mathbf{w}$).
- (Equations for a line) Given a point $P = (p_1, p_2, p_3)$ on a line and a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in the direction of the line, the following are parametric equations for the line:

  $$x(t) = p_1 + tv_1$$
  $$y(t) = p_2 + tv_2$$
  $$z(t) = p_3 + tv_3,$$

  which could also be written more succinctly as $\mathbf{r}(t) = \langle p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 \rangle$. (Parameterizations are not unique - they depend on the choices of P and $\mathbf{v}$.)
- (Distance from point to line) Given point S and a line with direction $\mathbf{v}$ and a point (any point) P, the distance from S to the line is $\frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$, where $\mathbf{PS}$ is the vector from P to S. If the line is given to you in parametric form, you can find a point on the line by plugging in any value of t, e.g., $t = 0$.
- (Equation for a plane) A plane is given by a normal vector $\langle A, B, C \rangle$ and a point (x_0, y_0, z_0) on the plane. The equation is then $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. One way to get the normal is to take the cross product of two vectors in the plane (that have the same initial point).

- (Line of intersection of two planes) Given two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, the vector $\mathbf{n}_1 \times \mathbf{n}_2$ points in the direction of the line of intersection of the two planes (assuming they intersect!). To get a point on this line, you can solve the system of two plane equations. Note that there will be 3 variables and 2 equations in this linear system, so you should just set one of the variables to a constant, e.g., $x = 0$, and solve for the other two. (Two planes are parallel – don't intersect – if they have parallel normals, i.e., $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}$.)

- (Distance from point to plane) Given point S and a plane with normal $\mathbf{n}$ and point P on the plane, the distance from S to the plane is $\frac{|\mathbf{PS}\cdot\mathbf{n}|}{|\mathbf{n}|}$.

- (Angle between planes) The angle between two planes is the angle between their normal vectors.

- (Position, velocity, and acceleration) If $\mathbf{r}(t)$ represents the position of a particle, then $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity and $\mathbf{a}(t) = \mathbf{v}'(t)$ is the acceleration. If you are given an initial value problem (e.g., $\mathbf{a}(t)$ and the values of $\mathbf{v}(t)$ and $\mathbf{r}(t)$ at some point), you integrate to work your way up to $\mathbf{r}(t)$, using the values of $\mathbf{v}(t)$ and $\mathbf{r}(t)$ to find the constants of integration.

- (Arclength) If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ gives a curve in $\mathbb{R}^3$, the arc length from the point at time $t = a$ to the point at time $t = b$ is

$$s = \int_a^b |\mathbf{v}(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$

- (Decomposing acceleration) Acceleration along a curve can be written as a linear combination of the tangent and normal directions. In particular:

$$\mathbf{a}(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t),$$

  where:

  - $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$,
  - $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ (this is not the easy way to compute this!),
  - $a_T = \mathbf{a} \cdot \mathbf{T}$, and
  - a_N can be found from the Pythagorean identity $|\mathbf{a}|^2 = a_T^2 + a_N^2$.

  If you need to compute all of these, the typical strategy is to compute $\mathbf{v}$ and $\mathbf{a}$, then compute $\mathbf{T}$ as above, then a_T, then a_N. Finally, you can find $\mathbf{N}$ using the fact that $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$.