MATH 676

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Finite element methods in scientific computing

Wolfgang Bangerth, Texas A&M University

Lecture 31.5:

Nonlinear problems

Part 1: Introduction

Nonlinear problems

Reality is nonlinear.
Linear equations are only approximations.

Nonlinear problems

Reality is nonlinear.
Linear equations are only approximations.

Linear equations typically assume that something is small:

- Poisson equation for displacement of a membrane
  Assumption: small displacement
- Stokes equation
  Assumption: slow flow, incompressible medium
- Maxwell equations
  Assumption: Small electromagnetic field strength

Fluid flow example, part 1

Consider the Stokes equations:

$$
\begin{aligned}
\rho \frac{\partial u}{\partial t} - \nu \Delta u + \nabla p &= f \\
\nabla \cdot u &= 0
\end{aligned}
$$

These equations are the *small-velocity approximation* of the nonlinear Navier-Stokes equations:

$$
\begin{aligned}
\rho \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) - \nu \Delta u + \nabla p &= f \\
\nabla \cdot u &= 0
\end{aligned}
$$

Fluid flow example, part 2

The Navier-Stokes equations

$$\begin{aligned} \rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)-\nu\Delta u+\nabla p &= f \\ \nabla\cdot u &= 0 \end{aligned}$$

are the *small-pressure approximation* of the variable-density Navier-Stokes equations:

$$\begin{aligned} \rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)-\nu\Delta u+\nabla p &= f \\ \nabla\cdot(\rho u) &= 0 \end{aligned}$$

Fluid flow example, part 3

The variable-density Navier-Stokes equations can be further generalized:

- The viscosity really depends on
  – pressure
  – strain rate
- Friction converts mechanical energy into heat
- Viscosity and density depend on temperature
- …

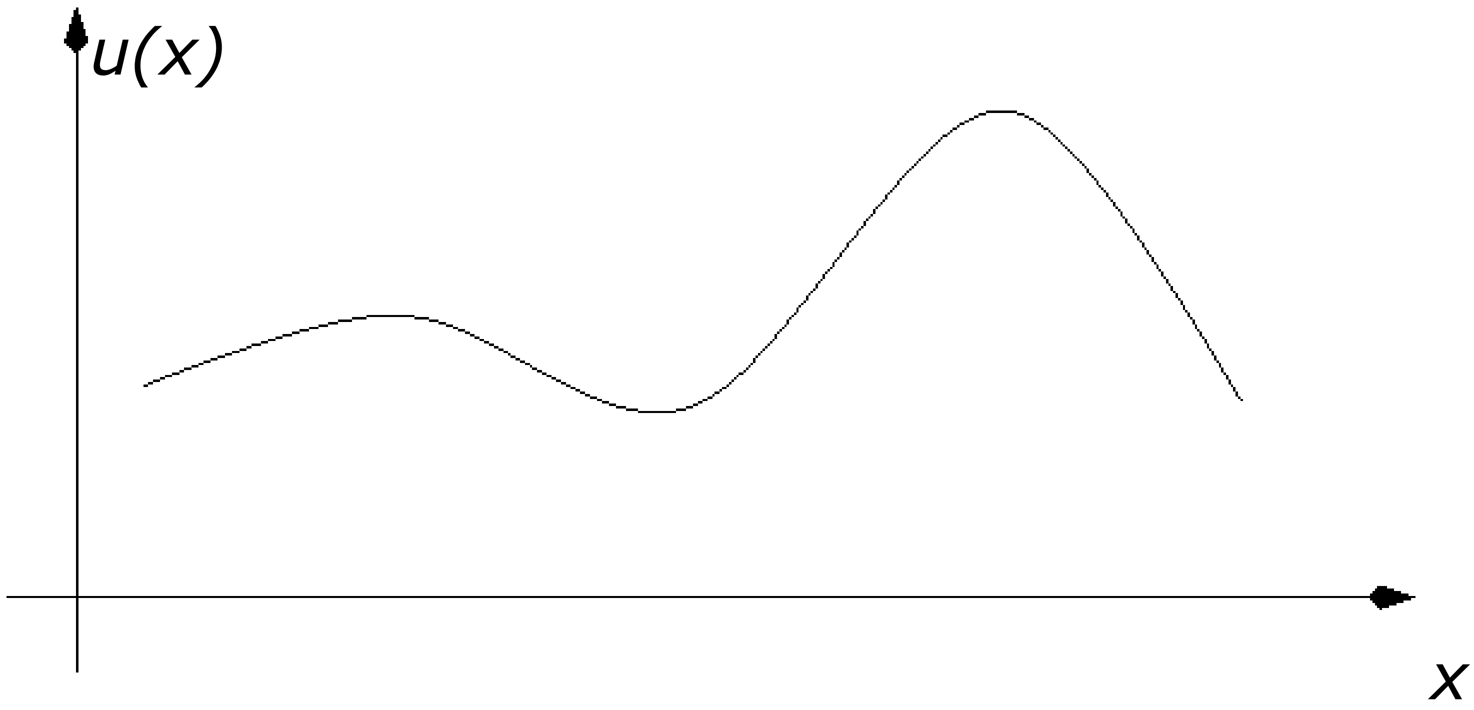
1d elastic deformation example

Consider a 1d rubber band:

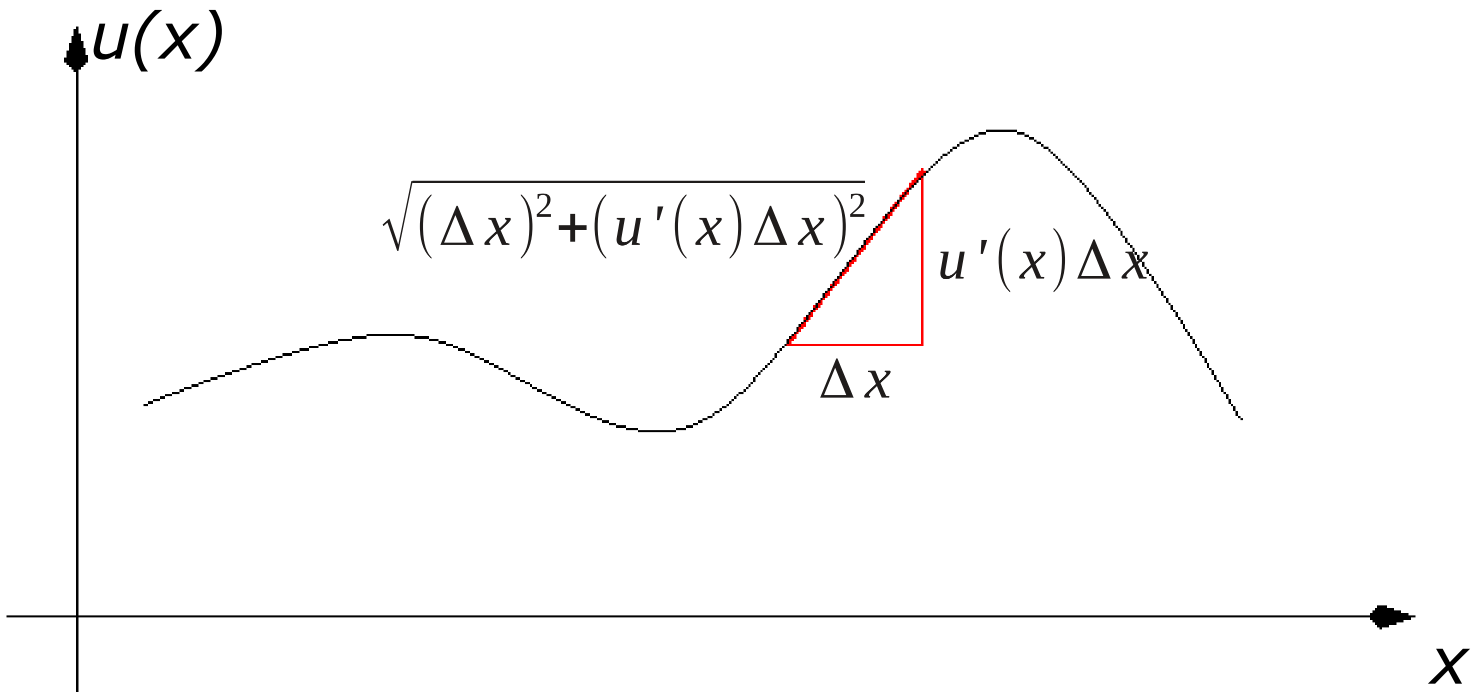
– Clamped at the ends

– Deformed perpendicularly by a force *f(x)*

– Leading to a perpendicular displacement *u(x)*



1d elastic deformation example



If a material is linearly elastic, then the energy stored in a deformation is proportional to its *elongation*:

$$E_{\text{deformation}}(u) = \lim_{\Delta x \to 0} \sum_{j=1,\Delta x=\frac{(b-a)}{N}}^{N} A\left(\sqrt{(\Delta x)^2+(u'(x_j)\Delta x)^2}-\Delta x\right)$$

$$= \int_a^b A\left(\sqrt{(dx)^2+(u'(x)dx)^2}-dx\right) = \int_a^b A\left(\sqrt{1+(u'(x))^2}-1\right)dx$$

1d elastic deformation example

The total energy is a sum of two terms:

– the deformation energy
– the work against an external force

$$
\begin{aligned}
E(u) &= E_{\text{deformation}} + E_{\text{potential}} \\
&= \int_a^b A\left(\sqrt{1+(u'(x))^2}-1\right)dx \;-\; \int_a^b f(x)u(x)\,dx \\
&= \int_a^b \left[A\left(\sqrt{1+(u'(x))^2}-1\right)-f(x)u(x)\right]dx
\end{aligned}
$$

1d elastic deformation example

We seek that displacement *u(x)* that minimizes the energy

$$E(u) = \int_a^b \left[A\left(\sqrt{1+(u'(x))^2}-1\right) - f(x)u(x) \right] dx$$

This is equivalent to finding that point *u(x)* for which every infinitesimal variation *εv(x)* leads to the same energy:

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[E(u+\epsilon v) - E(u)\right] = 0 \qquad \forall v \in H_0^1$$

In other words:

$$\int_a^b \left(A \frac{v'(x)u'(x)}{\sqrt{1+(u'(x))^2}} - v(x)f(x) \right) dx = 0 \qquad \forall v \in H_0^1$$

1d elastic deformation example

We seek that displacement *u(x)* that satisfies

$$\int_a^b \left(A \frac{v'(x) u'(x)}{\sqrt{1+(u'(x))^2}} - v(x) f(x) \right) dx = 0 \qquad \forall v \in H_0^1$$

The strong form of this equation is:

$$-\left(A \frac{u'(x)}{\sqrt{1+(u'(x))^2}} \right)' = f(x)$$

In multiple space dimensions, this generalizes to this:

$$-\nabla \cdot \left(A \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = f$$

This is often called the *minimal surface equation*.

Minimal surface vs. Poisson equation

Note: If the vertical displacement of the membrane is small and smooth, then

$$|\nabla u|^2 \ll 1$$

In this case, the (nonlinear) minimal surface equation

$$-\nabla \cdot \left(A \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = f$$

can be approximated by the (linear) Poisson equation:

$$-\nabla \cdot (A \nabla u) = f$$

What makes this complicated?

Start with the minimal surface equation

$$-\nabla \cdot \left(A \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = f$$

and its weak form:

$$\left(\nabla \phi, A \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = (\phi, f) \qquad \forall \phi \in H_0^1$$

Let's see what happens if we just discretize as always using

$$u_h(x) = \sum_j U_j \phi_j(x)$$

What makes this complicated?

Start with the minimal surface equation

$$-\nabla\cdot\left(A\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f$$

and discretize as always:

$$\left(\nabla\phi_i, A\frac{\nabla\sum_j U_j\phi_j}{\sqrt{1+|\nabla\sum_j U_j\phi_j|^2}}\right) = (\phi_i, f) \qquad \forall i=1\dots N$$

We can pull some coefficients and sums out:

$$\sum_j\left(\nabla\phi_i, A\frac{\nabla\phi_j}{\sqrt{1+|\sum_j U_j\nabla\phi_j|^2}}\right)U_j = (\phi_i, f) \qquad \forall i=1\dots N$$

This is a (potentially large) nonlinear system of equations!

What makes this complicated?

Start with the minimal surface equation

$$-\nabla \cdot \left(A \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = f$$

Discretizing as usual yields a system of nonlinear equations:

$$\sum_j \left(\nabla \phi_i, A \frac{\nabla \phi_j}{\sqrt{1+|\sum_j U_j \nabla \phi_j|^2}} \right) U_j = (\phi_i, f) \qquad \forall i=1 \ldots N$$

This could be written as

$$A(U)U = F$$

Problem: We don't know how to solve such systems directly. I.e., we know of no finite sequence of steps that yields the solution of general systems of nonlinear systems!

Nonlinear problems

In general: There is no finite algorithm to find simultaneous roots of a general system of nonlinear equations:

$$\begin{gathered} f_1(x_1,\dots,x_N)=0 \\ f_2(x_1,\dots,x_N)=0 \\ \vdots \\ f_N(x_1,\dots,x_N)=0 \end{gathered}$$

Or more concisely:

$$F(x)=0$$

However: Such algorithms exist for the linear case, e.g., Gaussian elimination.

Nonlinear problems

In fact: There is no finite algorithm to find a root of a single general nonlinear equation:

$$f(x) = 0$$

All algorithms for this kind of problem are *iterative*:

- Start with an initial guess x_0
- Compute a sequence of iterates $\{x_k\}$
- Hope (or prove) that $x_k \to x$ where x is a root of $f(.)$.

From here on: Consider only time-independent problems.

Approach to nonlinear problems

Goal: Find a "fixed point" *x* so that

$$f(x) = 0$$

Choose a function *g(x)* so that the solutions of

$$x = g(x)$$

are also roots of *f(x)*. Then iterate

$$x_{k+1} = g(x_k)$$

This iteration converges if *g* is a *contraction*.

Approach to nonlinear problems

Goal: Choose *g(x)* so that

$$x = g(x) \quad \Leftrightarrow \quad f(x)=0$$

Examples:

- "Picard iteration" (assume that *f(x)=p(x)x-h)*:

$$g(x) = \frac{1}{p(x)} h \quad \rightarrow \quad p(x_k)\, x_{k+1} = h$$

- Pseudo-timestepping:

$$g(x) = x \pm \Delta\tau f(x) \quad \rightarrow \quad \frac{x_{k+1}-x_k}{\Delta\tau} = \pm f(x_k)$$

- Newton's method

$$g(x) = x - \frac{f(x)}{f'(x)} \quad \rightarrow \quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Application to the minimal surface equation

Goal: Solve

$$-\nabla \cdot \left(A \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = f$$

Picard iteration: Repeatedly solve

$$-\nabla \cdot \left(A \frac{\nabla u_{k+1}}{\sqrt{1+|\nabla u_k|^2}} \right) = f$$

or in weak form:

$$\left(\nabla \phi, \frac{A}{\sqrt{1+|\nabla u_k|^2}} \nabla u_{k+1} \right) = (\phi, f) \qquad \forall \phi \in H_0^1$$

This is a linear PDE in u_{k+1}. We know how to do this.

Application to the minimal surface equation

Goal: Solve

$$-\nabla\cdot\left(A\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f$$

Picard iteration: Repeatedly solve

$$\left(\nabla\phi,\frac{A}{\sqrt{1+|\nabla u_k|^2}}\nabla u_{k+1}\right) = (\phi,f) \qquad \forall\phi\in H_0^1$$

Pros and cons:

- This is like the Poisson equation with a spatially varying coefficient (like step-6) → SPD matrix, easy
- Converges frequently
- Picard iteration typically converges rather slowly

Application to the minimal surface equation

Goal: Solve

$$-\nabla\cdot\left(A\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f$$

Pseudo-timestepping: Iterate to $\tau\to\infty$ the equation

$$\frac{\partial u(\tau)}{\partial\tau}-\nabla\cdot\left(A\frac{\nabla u(\tau)}{\sqrt{1+|\nabla u(\tau)|^2}}\right) = f$$

For example using the explicit Euler method:

$$\frac{u_{k+1}-u_k}{\Delta\tau}-\nabla\cdot\left(A\frac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}}\right) = f$$

Application to the minimal surface equation

Goal: Solve

$$-\nabla\cdot\left(A\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f$$

Pseudo-timestepping: Iterate to $\tau\rightarrow\infty$ the equation

$$\frac{\partial u(\tau)}{\partial\tau}-\nabla\cdot\left(A\frac{\nabla u(\tau)}{\sqrt{1+|\nabla u(\tau)|^2}}\right) = f$$

Alternatively (and better): Semi-implicit Euler method...

$$\frac{u_{k+1}-u_k}{\Delta\tau}-\nabla\cdot\left(A\frac{\nabla u_{k+1}}{\sqrt{1+|\nabla u_k|^2}}\right) = f$$

...or some higher order time stepping method.

Application to the minimal surface equation

Goal: Solve

$$-\nabla \cdot \left(A \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = f$$

Pseudo-timestepping: Semi-implicit Euler method

$$\frac{u_{k+1}-u_k}{\Delta \tau} - \nabla \cdot \left(A \frac{\nabla u_{k+1}}{\sqrt{1+|\nabla u_k|^2}} \right) = f$$

Pros and cons:

- Pseudo-timestepping converges almost always
- Easy to implement (it's just a heat equation)
- With implicit method, can make time step larger+larger
- Often takes *many many* time steps

Application to the minimal surface equation

Goal: Solve

$$-\nabla\cdot\left(A\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f$$

Newton's method: Consider the residual

$$R(u) = f+\nabla\cdot\left(A\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

Solve *R(u)=0* by using the iteration

$$u_{k+1} = u_k-[R'(u_k)]^{-1}R(u_k)$$

or equivalently:

$$[R'(u_k)]\,\delta u_k = -R(u_k), \qquad u_{k+1} = u_k+\delta u_k$$

Application to the minimal surface equation

Goal: Solve

$$-\nabla\cdot\left(A\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f$$

Newton's method: Iterate on

$$[R'(u_k)]\,\delta u_k = -R(u_k), \qquad u_{k+1} = u_k + \delta u_k$$

Here, this means:

$$-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u_k|^2}}\nabla\delta u_k\right)+\nabla\cdot\left(A\frac{\nabla u_k\cdot\nabla\delta u_k}{\left(1+|\nabla u_k|^2\right)^{3/2}}\nabla u_k\right) = f+\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u_k|^2}}\nabla u_k\right)$$

This is in fact a symmetric and positive definite problem.

Application to the minimal surface equation

Goal: Solve

$$-\nabla \cdot \left(A \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = f$$

Newton's method: Iterate on

$$[R'(u_k)]\,\delta u_k = -R(u_k), \qquad u_{k+1} = u_k + \delta u_k$$

Pros and cons:

- Rapid, quadratic convergence
- Only converges when started close enough to the solution
- Operator has different structure than in Picard

Summary for nonlinear problems

- Nonlinear (stationary) PDEs are difficult because there are no direct algorithms for nonlinear systems of equations

- In the context of PDEs, we typically use one of three classes of methods:

  – Picard iteration
    . converges frequently, but slowly

  – Pseudo-timestepping
    . converges most reliably, but slowly

  – Newton iteration
    . does not always converge
    . if it converges, then very rapidly

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