Finite element methods in scientific computing

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Lecture 3.93:

The ideas behind the finite element method

Part 4: Finding an approximation

Two fundamental questions

Question 1: What is a good way to *approximate* functions that requires only finitely much data/computation?

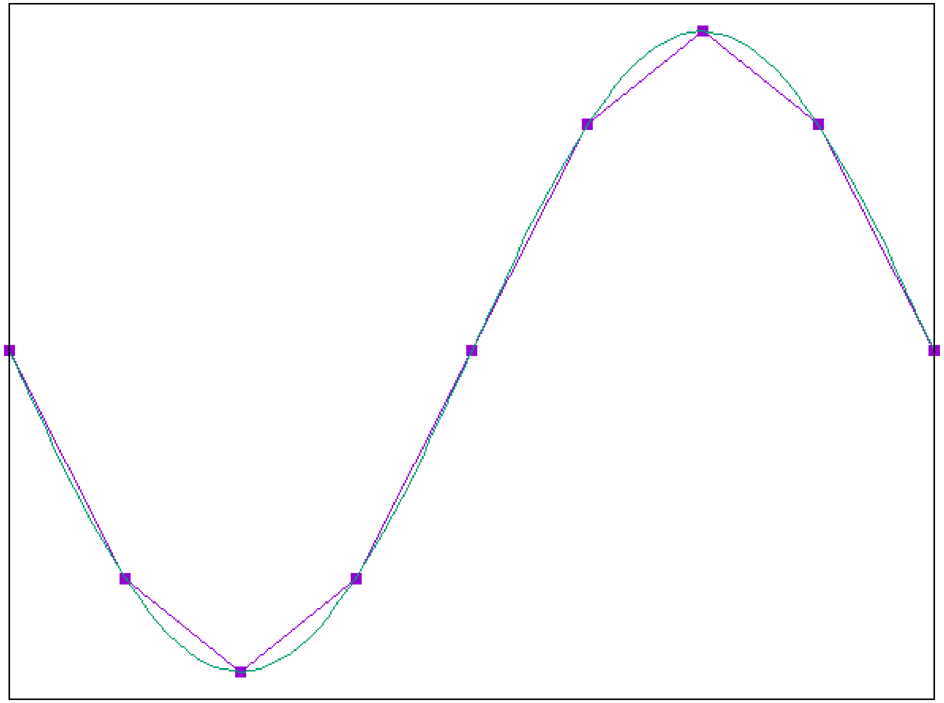
Question 2: How do we *find an approximation of the solution* of a PDE *without knowing the solution itself*?

How to find the approximation

Let us assume the following situation for now:

- We are in 1d
- We want to solve

$$-\frac{d^2}{dx^2}u(x) = f(x)$$



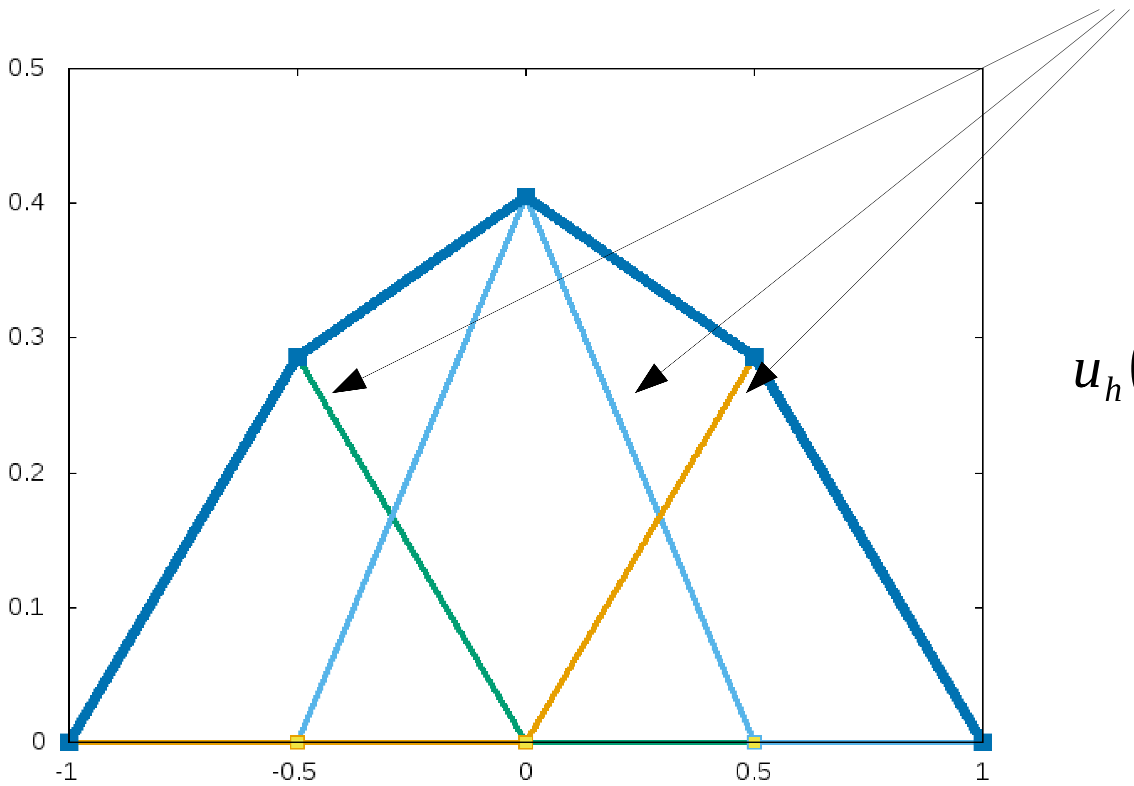
- We seek a piecewise linear approximation of the solution *u(x)*
- We will call the approximation $u_h(x)$

How to find the approximation

A little bit of mathematical abstraction:

Every piecewise linear function can be written in the form

$$u_h(x) = \sum_{j=1}^{N} \underbrace{U_j \varphi_j(x)}$$



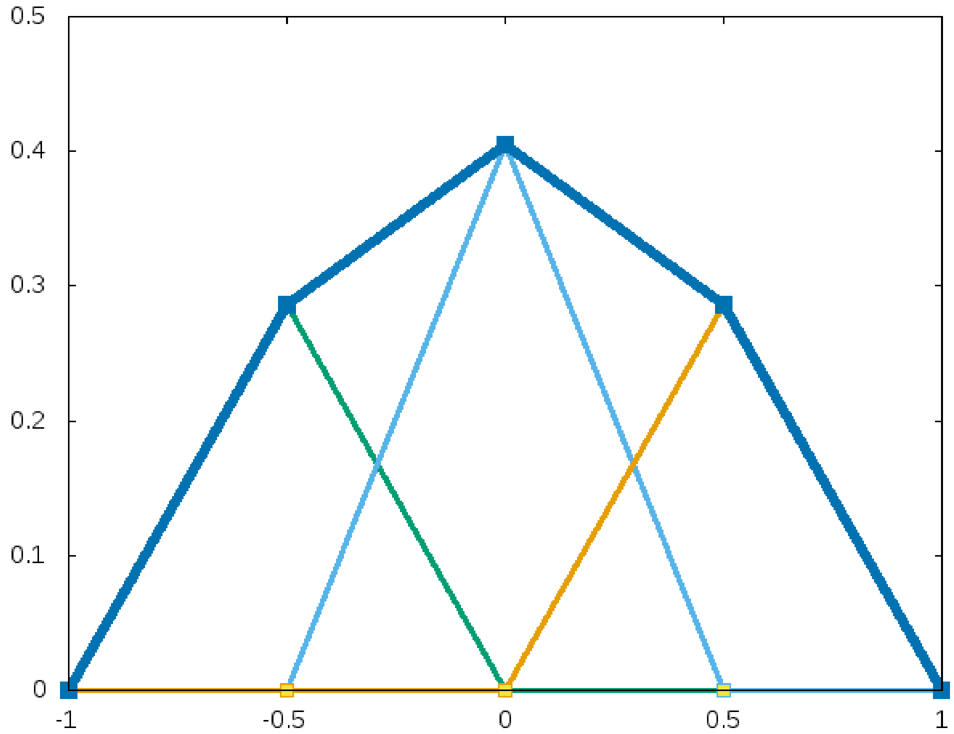
$$u_h(x) = \text{dark blue}$$
$$= \text{green} + \text{light blue} + \text{orange}$$

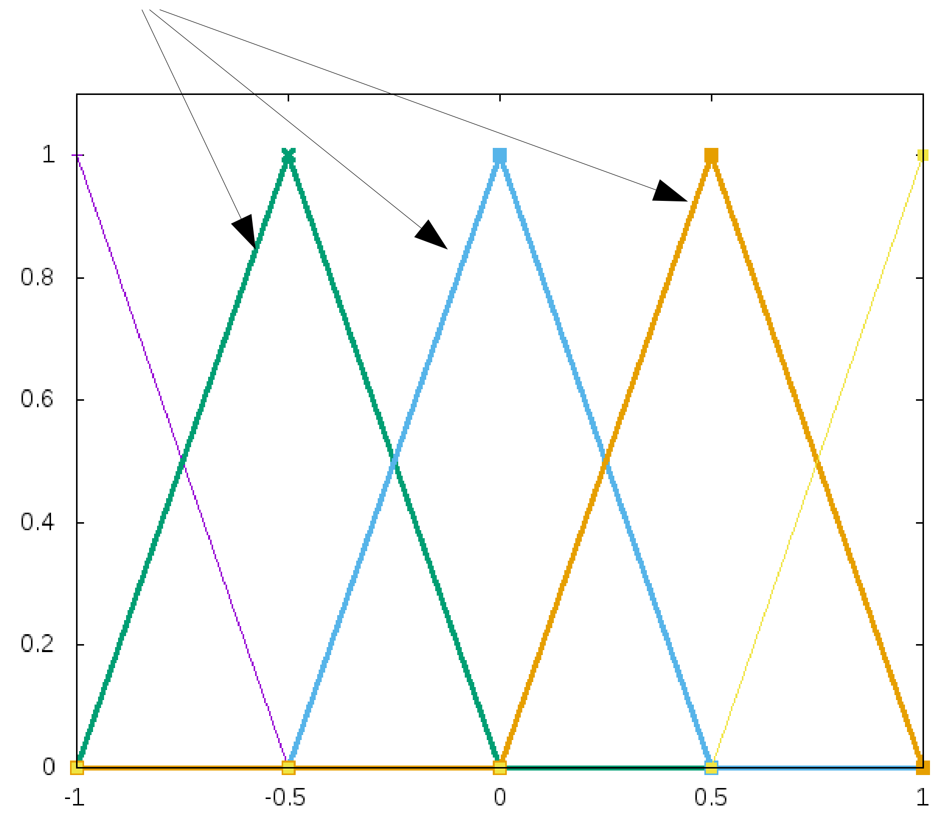
How to find the approximation

A little bit of mathematical abstraction:

Every piecewise linear function can be written in the form

$$u_h(x) = \sum_{j=1}^{N} U_j \varphi_j(x)$$





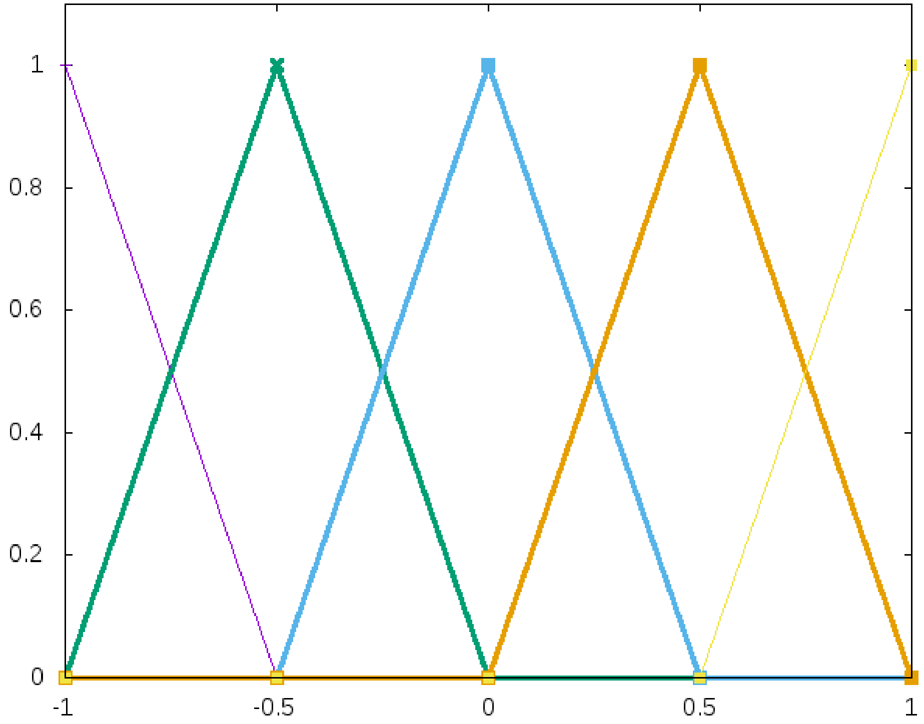
How to find the approximation

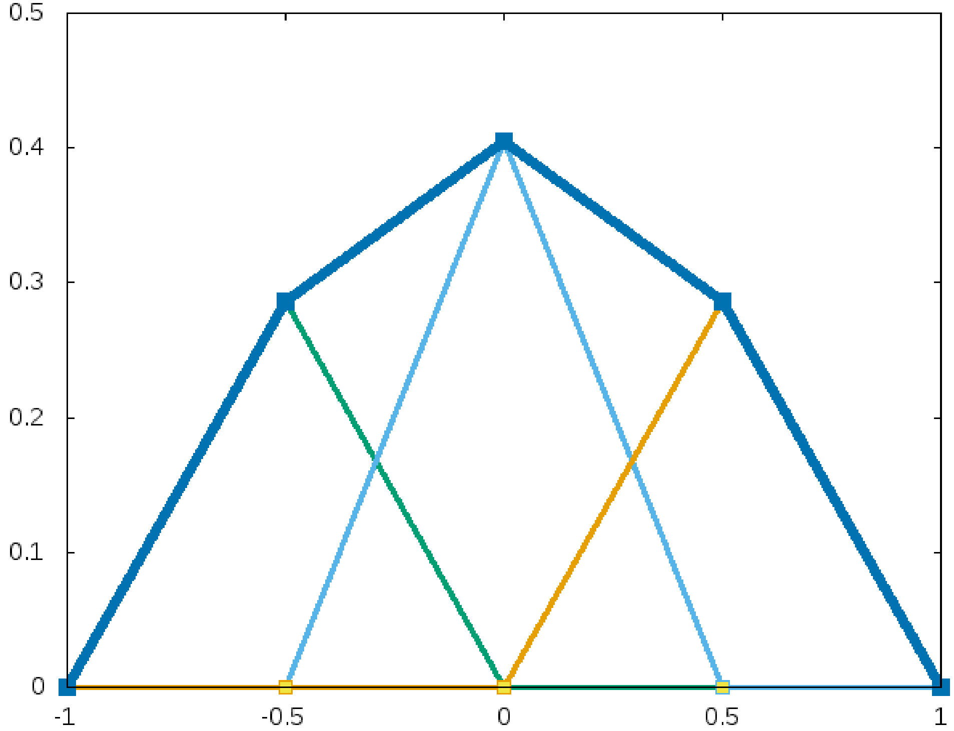
A little bit of mathematical abstraction:

In the example at the bottom right, we have

$$u_h(x) = 0 \cdot \varphi_1(x) + 0.3 \cdot \varphi_2(x) + 0.4 \cdot \varphi_3(x) + 0.3 \cdot \varphi_4(x) + 0 \cdot \varphi_5(x)$$

$$= \sum_{j=1}^{N} U_j \varphi_j(x) \qquad \text{with} \quad U = [0, 0.3, 0.4, 0.3, 0]^T$$





How to find the approximation

A little bit of mathematical abstraction:

Every piecewise linear function can be written in the form

$$u_h(x) = \sum_{j=1}^{N} U_j \varphi_j(x)$$

In other words: To know $u_h(x)$, we only need to know the (finitely many) coefficients U_j.

Two fundamental questions

Question 2: How do we *find an approximation of the solution* of a PDE *without knowing the solution itself*?

Equivalently: How to find the coefficients U_j that define the approximation u_h?

Answer: We need to use the PDE!

How to find the approximation

Idea 1:

Take the form

$$u_h(x) = \sum_{j=1}^{N} U_j \varphi_j(x)$$

and put it into the differential equation:

$$-\frac{d^2}{dx^2} u_h(x) = f(x)$$

This does not work:

- Second derivatives $\frac{d^2}{dx^2} u_h(x)$ are zero on each interval
- Second derivatives are not defined at the “node points”

$\rightarrow -\frac{d^2}{dx^2} u_h(x)$ can not equal $f(x)$!

How to find the approximation

Idea 2: Use the mathematical theory of "weak solutions"

Starting point for this theory: When we say that we want two functions *g(x)* and *h(x)* to be equal,

$$g = h$$

what do we actually mean by that?

- That they are equal for every *x*?
- That they are equal for almost every *x*?
- ...?

The problem arises because we only know how to compare *numbers*, but we now need to compare functions!

Weak solutions

A solution: Turn the equation

$$g = h$$

into an (infinite) number of comparisons of numbers.

In particular:

- We say that g equals h if *everything we can measure about g and h is equal*.

Weak solutions

A solution: Turn the equation

$$g = h$$

into an (infinite) number of comparisons of numbers.

In particular: We say that g equals h if

$$\begin{aligned} F_1[g] &= F_1[h] \\ F_2[g] &= F_2[h] \\ F_3[g] &= F_3[h] \\ &\dots \end{aligned}$$

for an infinite number of "measurements" $F_i[.]$.

Mathematically: Measurement = "linear functional" – something that takes a function as argument and returns a number.

Weak solutions

A solution: Turn the equation

$$g = h$$

into an (infinite) number of comparisons of numbers.

In particular:

- We say that g equals h if *everything we can measure about g and h is equal*.

- What we can measure depends on the set of functions we consider.

Weak solutions

What set of functions do we have: We want to compare

$$-\Delta u = f$$

So:

- The equation describes (for example) steady state concentrations *u* and concentration source densities *f*
- So we are comparing concentration source densities
- Point values of source densities are meaningless because they do not affect concentrations
- We can measure volume integrals/averages of *f*
- We can measure weighted integrals/averages of *f*

Weak solutions

Example 1: We say that g equals h on the interval (0,1) if

$$\int_0^1 g(x)\,dx = \int_0^1 h(x)\,dx$$
$$\int_0^1 x\,g(x)\,dx = \int_0^1 x\,h(x)\,dx$$
$$\int_0^1 x^2 g(x)\,dx = \int_0^1 x^2 h(x)\,dx$$
$$\ldots$$

Weak solutions

Example 2: We say that g equals h on the interval (0,1) if

$$\begin{aligned}
\int_0^1 g(x)\,dx &= \int_0^1 h(x)\,dx \\
\int_0^1 \sin(\pi x)g(x)\,dx &= \int_0^1 \sin(\pi x)h(x)\,dx \\
\int_0^1 \cos(\pi x)g(x)\,dx &= \int_0^1 \cos(\pi x)h(x)\,dx \\
\int_0^1 \sin(2\pi x)g(x)\,dx &= \int_0^1 \sin(2\pi x)h(x)\,dx \\
\int_0^1 \cos(2\pi x)g(x)\,dx &= \int_0^1 \cos(2\pi x)h(x)\,dx \\
&\dots
\end{aligned}$$

Weak solutions

For differential equations: We say a function *u(x)* is a "weak solution" of the PDE if

$$F_1\left[-\frac{d^2}{dx^2}u\right] = F_1[f]$$

$$F_2\left[-\frac{d^2}{dx^2}u\right] = F_2[f]$$

$$F_3\left[-\frac{d^2}{dx^2}u\right] = F_3[f]$$

$$\ldots$$

where we choose $F_k[g] = \int_\Omega \varphi_k(x)\, g(x)\, dx$ for an infinite set of functions $\varphi_k(x)$

Weak solutions

Put differently: We say a function *u(x)* is a "weak solution" of the PDE if the equation

$$\int_\Omega \varphi(x)\left[-\frac{d^2}{dx^2}u(x)\right]dx = \int_\Omega \varphi(x) f(x)\, dx$$

holds "for all *test functions* functions $\varphi(x)$".

In mathematical notation:

$$\int_\Omega \varphi(x)\left[-\frac{d^2}{dx^2}u(x)\right]dx = \int_\Omega \varphi(x) f(x)\, dx \qquad \forall \varphi$$

Weak solutions

We would like to treat solution and test functions the same:

We can achieve this by integrating by parts:

$$\int_\Omega \varphi(x)\left[-\frac{d^2}{dx^2}u(x)\right]dx = \int_\Omega \left[\frac{d}{dx}\varphi(x)\right]\left[\frac{d}{dx}u(x)\right]dx + \text{boundary terms}$$

For now, we will ignore boundary terms.

In mathematical notation: *u(x)* is a solution if

$$\int_\Omega \left[\frac{d}{dx}\varphi(x)\right]\left[\frac{d}{dx}u(x)\right]dx = \int_\Omega \varphi(x) f(x)\,dx \qquad \forall \varphi$$

A side note

Remark: The following two problems are equivalent:

$$\int_\Omega \left[\frac{d}{dx}\varphi_1(x)\right]\left[\frac{d}{dx}u(x)\right]dx = \int_\Omega \varphi_1(x) f(x)\,dx$$
$$\int_\Omega \left[\frac{d}{dx}\varphi_2(x)\right]\left[\frac{d}{dx}u(x)\right]dx = \int_\Omega \varphi_2(x) f(x)\,dx$$
$$\dots$$

and

$$\int_\Omega \left[\frac{d}{dx}\varphi(x)\right]\left[\frac{d}{dx}u(x)\right]dx = \int_\Omega \varphi(x) f(x)\,dx \qquad \forall \varphi$$

This is because “every function” can be expressed as

$$\varphi(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x)$$

How to find the approximation

Idea 2: Use the mathematical theory of "weak solutions"

We know that the exact solution satisfies the equality

$$\int_\Omega \left[\frac{d}{dx}\varphi(x)\right]\left[\frac{d}{dx}u(x)\right]dx = \int_\Omega \varphi(x) f(x)\, dx \qquad \forall \varphi$$

So we could try to find an approximate solution u_h that satisfies

$$\int_\Omega \left[\frac{d}{dx}\varphi(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx = \int_\Omega \varphi(x) f(x)\, dx \qquad \forall \varphi$$

How to find the approximation

Idea 2: Use the mathematical theory of "weak solutions" to find an approximate solution:

We seek

$$u_h(x) = \sum_{j=1}^{N} U_j \varphi_j(x)$$

so that

$$\int_\Omega \left[\frac{d}{dx}\varphi(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx = \int_\Omega \varphi(x) f(x) dx \qquad \forall \varphi$$

Pro: Only first derivatives on u_h

Con: Only N unknowns U_j, but *infinitely many equations*!

How to find the approximation

Idea 3: Restrict the set of "test functions" in the "weak formulation" to find an approximate solution:

We seek

$$u_h(x) = \sum_{j=1}^{N} U_j \varphi_j(x)$$

so that

$$\int_\Omega \left[\frac{d}{dx}\varphi_h(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx = \int_\Omega \varphi_h(x) f(x)\, dx \qquad \forall \varphi_h = \sum_{k=1}^{N} c_k \varphi_k$$

This is equivalent to *N* equations for *N* unknowns!

This is called the *Galerkin Method*.

How to find the approximation

Equivalently, the “Galerkin method” reads:

Find

$$u_h(x) = \sum_{j=1}^{N} U_j \varphi_j(x)$$

so that

$$\int_\Omega \left[\frac{d}{dx}\varphi_1(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx = \int_\Omega \varphi_1(x) f(x)\, dx$$

$$\int_\Omega \left[\frac{d}{dx}\varphi_2(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx = \int_\Omega \varphi_2(x) f(x)\, dx$$

$$\cdots$$

$$\int_\Omega \left[\frac{d}{dx}\varphi_N(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx = \int_\Omega \varphi_N(x) f(x)\, dx$$

How to find the approximation

The *N* equations form a *linear system of equations*:

To see this, insert

$$u_h(x) = \sum_{j=1}^{N} U_j \varphi_j(x)$$

into the equations:

$$\begin{aligned}
\int_\Omega \left[\frac{d}{dx}\varphi_1(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx &= \int_\Omega \varphi_1(x) f(x)\,dx \\
\int_\Omega \left[\frac{d}{dx}\varphi_2(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx &= \int_\Omega \varphi_2(x) f(x)\,dx \\
&\cdots \\
\int_\Omega \left[\frac{d}{dx}\varphi_N(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx &= \int_\Omega \varphi_N(x) f(x)\,dx
\end{aligned}$$

How to find the approximation

The *N* equations form a *linear system of equations*:

This results in:

$$\int_\Omega \left[\frac{d}{dx}\varphi_1(x)\right]\left[\frac{d}{dx}\sum_{j=1}^N U_j\varphi_j(x)\right]dx = \int_\Omega \varphi_1(x) f(x)\,dx$$

$$\int_\Omega \left[\frac{d}{dx}\varphi_2(x)\right]\left[\frac{d}{dx}\sum_{j=1}^N U_j\varphi_j(x)\right]dx = \int_\Omega \varphi_2(x) f(x)\,dx$$

$$\cdots$$

$$\int_\Omega \left[\frac{d}{dx}\varphi_N(x)\right]\left[\frac{d}{dx}\sum_{j=1}^N U_j\varphi_j(x)\right]dx = \int_\Omega \varphi_N(x) f(x)\,dx$$

How to find the approximation

The *N* equations form a *linear system of equations*:

Move the sum out of the integral:

$$\sum_{j=1}^{N} \int_\Omega \left[\frac{d}{dx}\varphi_1(x)\right]\left[\frac{d}{dx}U_j\varphi_j(x)\right]dx = \int_\Omega \varphi_1(x) f(x)\,dx$$

$$\sum_{j=1}^{N} \int_\Omega \left[\frac{d}{dx}\varphi_2(x)\right]\left[\frac{d}{dx}U_j\varphi_j(x)\right]dx = \int_\Omega \varphi_2(x) f(x)\,dx$$

$$\cdots$$

$$\sum_{j=1}^{N} \int_\Omega \left[\frac{d}{dx}\varphi_N(x)\right]\left[\frac{d}{dx}U_j\varphi_j(x)\right]dx = \int_\Omega \varphi_N(x) f(x)\,dx$$

How to find the approximation

The *N* equations form a *linear system of equations*:

Move the coefficients out of the integral:

$$\sum_{j=1}^{N} \left\{ \int_\Omega \left[\frac{d}{dx} \varphi_1(x) \right] \left[\frac{d}{dx} \varphi_j(x) \right] dx \right\} U_j = \int_\Omega \varphi_1(x) f(x) dx$$

$$\sum_{j=1}^{N} \left\{ \int_\Omega \left[\frac{d}{dx} \varphi_2(x) \right] \left[\frac{d}{dx} \varphi_j(x) \right] dx \right\} U_j = \int_\Omega \varphi_2(x) f(x) dx$$

$$\cdots$$

$$\sum_{j=1}^{N} \left\{ \int_\Omega \left[\frac{d}{dx} \varphi_N(x) \right] \left[\frac{d}{dx} \varphi_j(x) \right] dx \right\} U_j = \int_\Omega \varphi_N(x) f(x) dx$$

Note: We can think of the terms in braces as entries of a matrix A_{ij}! Same for the right hand side

How to find the approximation

The *N* equations form a *linear system of equations*:

That is:

$$\begin{aligned}
\sum_{j=1}^{N} A_{1j} U_j &= F_1 \\
\sum_{j=1}^{N} A_{2j} U_j &= F_2 \\
&\dots \\
\sum_{j=1}^{N} A_{Nj} U_j &= F_N
\end{aligned}$$

$$\text{with } A_{ij} := \int_\Omega \left[\frac{d}{dx}\varphi_i(x)\right]\left[\frac{d}{dx}\varphi_j(x)\right] dx$$

$$\text{and } F_i := \int_\Omega \varphi_i(x) f(x)\, dx$$

How to find the approximation

The *N* equations form a *linear system of equations*:

Or in shorter notation:

$$AU = F$$

$$\text{with } A_{ij} := \int_\Omega \left[\frac{d}{dx}\varphi_i(x)\right]\left[\frac{d}{dx}\varphi_j(x)\right]dx$$

$$\text{and } F_i := \int_\Omega \varphi_i(x) f(x)\, dx$$

Conclusion: To find u_h, we need U, for which we only need to solve a linear system!

A word on notation

We typically use the following abbreviated notation:

$$(g, h)_\Omega := \int_\Omega g(x)\, h(x)\, dx$$

We can then re-write the problem

$$\int_\Omega \left[\frac{d}{dx}\varphi_h(x)\right]\left[\frac{d}{dx}u_h(x)\right] dx = \int_\Omega \varphi_h(x) f(x)\, dx \qquad \forall \varphi_h = \sum_{k=1}^N c_k \varphi_k$$

as follows:

$$\left(\frac{d}{dx}\varphi_h, \frac{d}{dx}u_h\right) = (\varphi_h, f)_\Omega \qquad \forall \varphi_h = \sum_{k=1}^N c_k \varphi_k$$

A word on notation

Similarly:

$$\begin{aligned}
\int_\Omega \left[\frac{d}{dx}\varphi_1(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx &= \int_\Omega \varphi_1(x) f(x)\,dx \\
\int_\Omega \left[\frac{d}{dx}\varphi_2(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx &= \int_\Omega \varphi_2(x) f(x)\,dx \\
&\dots \\
\int_\Omega \left[\frac{d}{dx}\varphi_N(x)\right]\left[\frac{d}{dx}u_h(x)\right]dx &= \int_\Omega \varphi_N(x) f(x)\,dx
\end{aligned}$$

is the same as:

$$\begin{aligned}
\left(\frac{d}{dx}\varphi_1(x), \frac{d}{dx}u_h(x)\right) &= \left(\varphi_1(x), f(x)\right) \\
\left(\frac{d}{dx}\varphi_2(x), \frac{d}{dx}u_h(x)\right) &= \left(\varphi_2(x), f(x)\right) \\
&\dots \\
\left(\frac{d}{dx}\varphi_N(x), \frac{d}{dx}u_h(x)\right) &= \left(\varphi_N(x), f(x)\right)
\end{aligned}$$

A word on notation

We then often write the linear system to solve like this:

$$AU = F$$

$$\text{with } A_{ij} := \left(\left[\frac{d}{dx} \varphi_i \right], \left[\frac{d}{dx} \varphi_j \right] \right)$$

$$\text{and } F_i := \left(\varphi_i, f \right)$$

How to find the approximation

Similarly, in higher dimensions this look as follows:

Start with

$$-\Delta u(\vec{x}) = f(\vec{x})$$

Multiply by a test function, integrate:

$$\int_\Omega \varphi(\vec{x})[-\Delta u(\vec{x})]\,dx = \int_\Omega \varphi(\vec{x}) f(\vec{x})\,dx$$

Then integrate by parts on the left hand side:

$$\int_\Omega \varphi(\vec{x})[-\Delta u(\vec{x})]\,dx = \int_\Omega [\nabla \varphi(\vec{x})]\cdot[\nabla u(\vec{x})]\,dx + \text{boundary terms}$$

How to find the approximation

Similarly, in higher dimensions this look as follows:

We then seek

$$u_h(\vec{x}) = \sum_{j=1}^{N} U_j \varphi_j(\vec{x})$$

so that

$$\int_\Omega [\nabla \varphi_h(\vec{x})] \cdot [\nabla u_h(\vec{x})] dx = \int_\Omega \varphi_h(\vec{x}) f(\vec{x}) dx \qquad \forall \varphi_h = \sum_{k=1}^{N} c_k \varphi_k$$

Or, in shorthand notation:

$$(\nabla \varphi_h, \nabla u_h) = (\varphi_h, f) \qquad \forall \varphi_h = \sum_{k=1}^{N} c_k \varphi_k$$

More questions

For this method to be useful, we need to ask more questions:

Question 3: Is the approximation u_h so defined "close" to the exact solution u?

Question 4: Does u_h "converge" towards u in some useful sense?

Question 5: What is the computational effort to reach a certain accuracy? Optimality?

These are all non-trivial mathematical questions left for later lectures.

Finite element methods in scientific computing

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