Finite element methods in scientific computing

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Lecture 3.92:

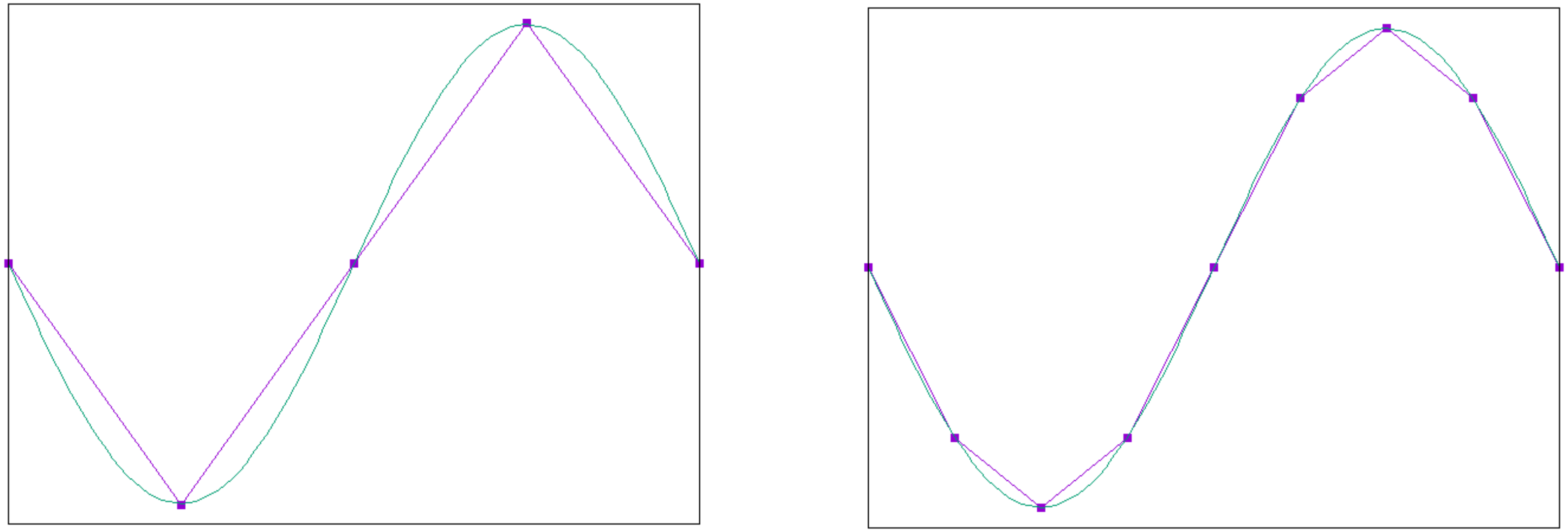
The ideas behind the finite element method

Part 3: Piecewise polynomial approximation in 2d/3d

Piecewise polynomial approximation in 1d

In 1d:

- Split the domain $\Omega \subset R$ of a function into intervals
- Define a piecewise function: linear on each interval, continuous at interval boundaries
- More sub-intervals → better approximation



Green: The function $f(x)$ we want to approximate.

Purple: The piecewise linear approximant $f_h(x)$.

Piecewise polynomial approximation in 2d

In 1d:

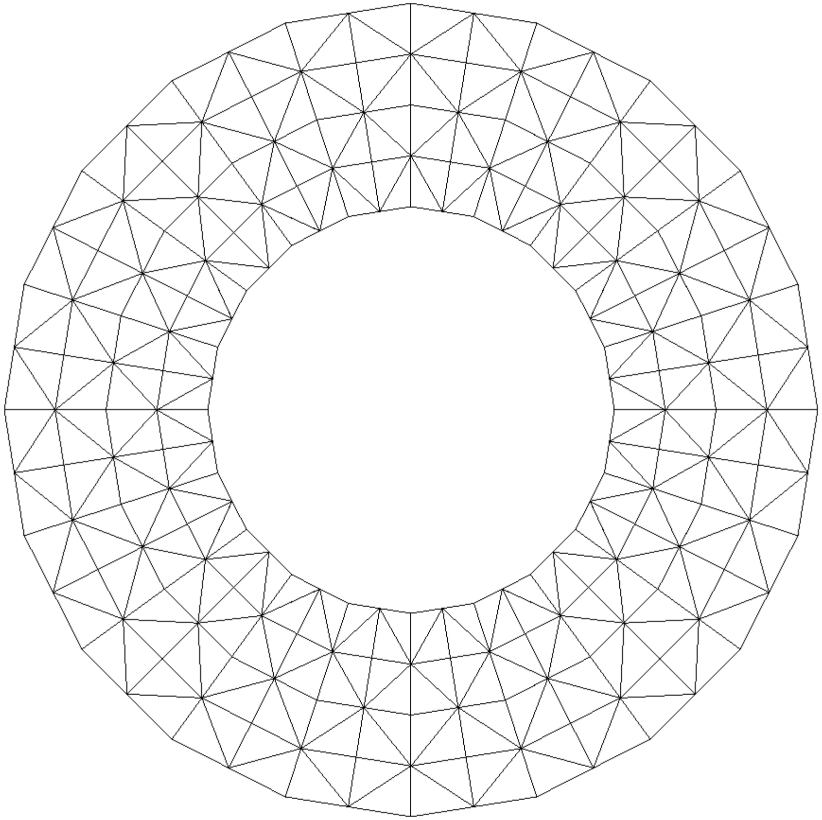
- Split the domain $\Omega \subset R$ of a function into intervals
- Define a piecewise function: linear on each interval, continuous at interval boundaries
- More sub-intervals → better approximation

In 2d:

- Split the domain $\Omega \subset R^2$ of a function into “cells”
- Define a piecewise function: linear on each cell, continuous at cell boundaries
- Smaller cells → better approximation

Piecewise polynomial approximation in 2d

Example with triangles:
Step 1: Subdivide the domain into triangular “cells”.

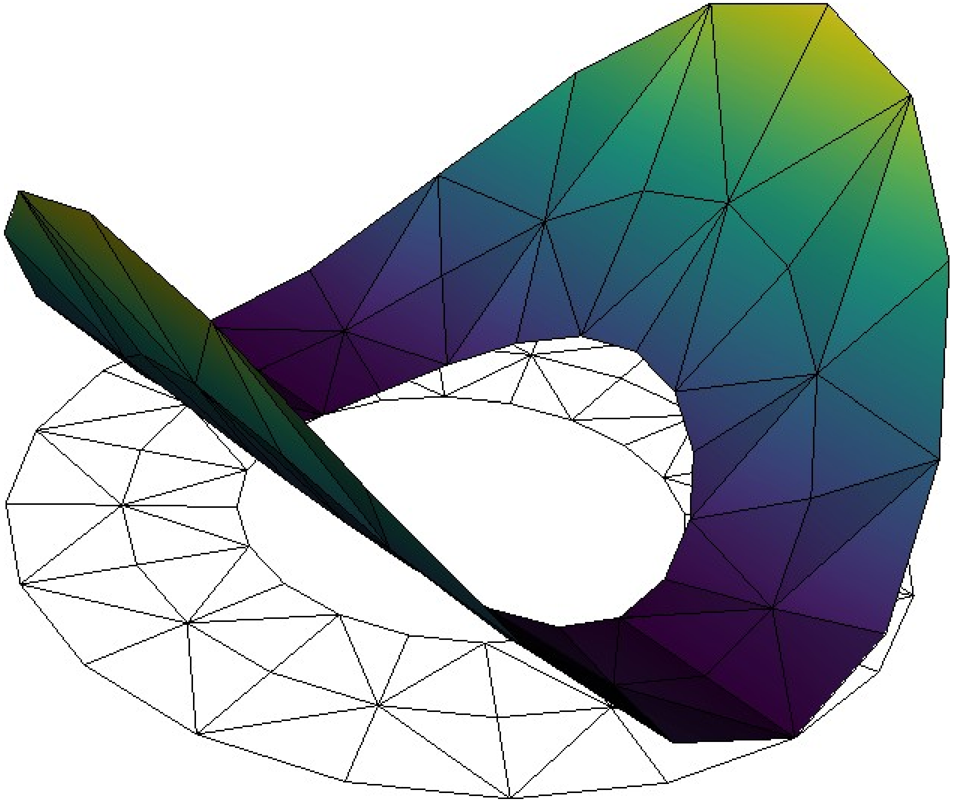


We call the collection of cells the “mesh”.

Piecewise polynomial approximation in 2d

Example with triangles:

Step 2: Represent functions as *piecewise polynomials*.

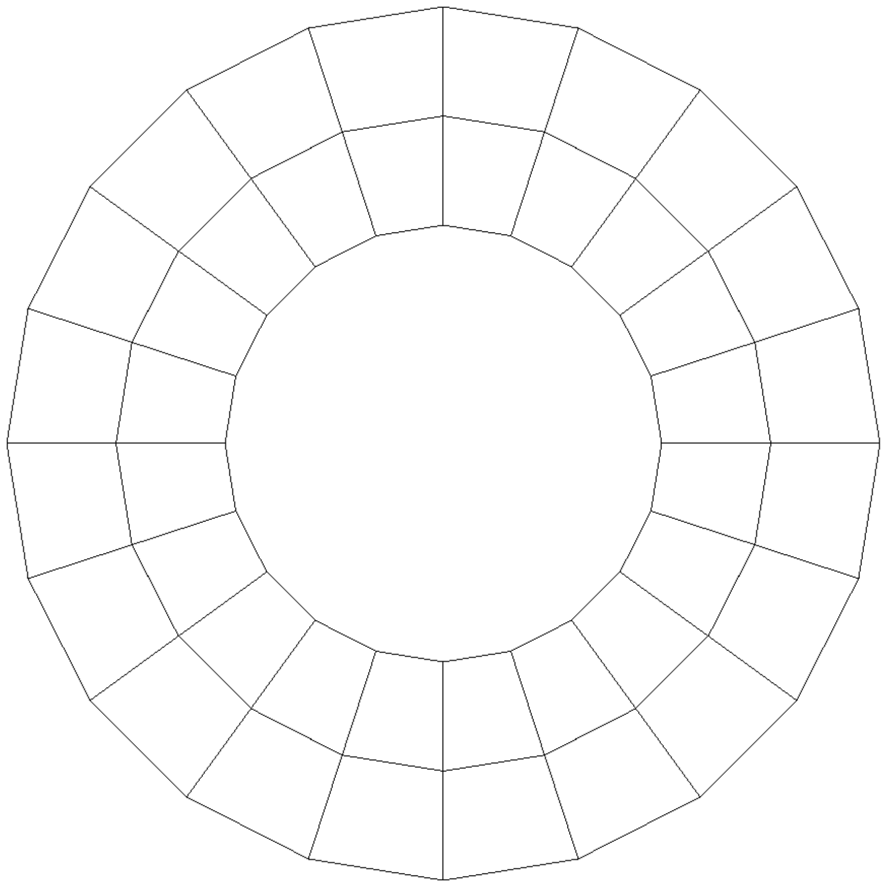


The function is linear on each triangle.

Piecewise polynomial approximation in 2d

Example with quadrilaterals:

Step 1: Subdivide the domain into *quadrilateral* “cells”.

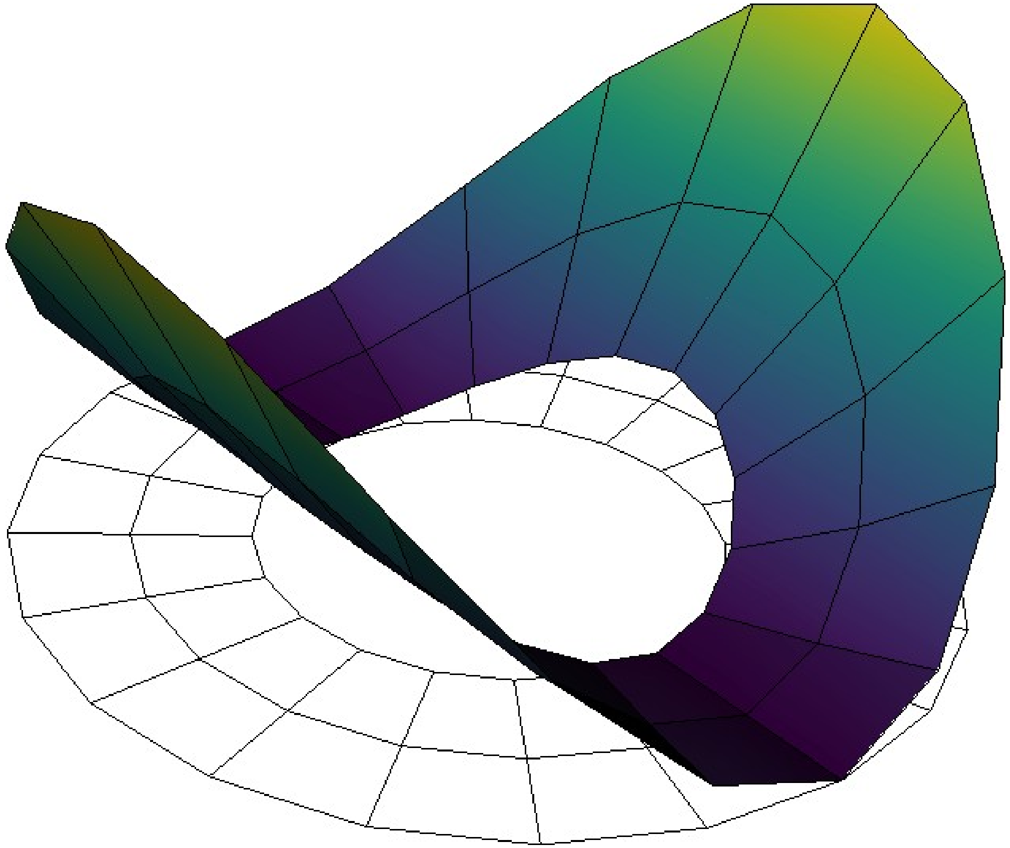


We call the collection of cells the “mesh”.

Piecewise polynomial approximation in 2d

Example with quadrilaterals:

Step 2: Represent functions as *piecewise polynomials*.



The function is bi-linear on each quadrilateral.

Piecewise polynomial approximation in 3d

In 2d:

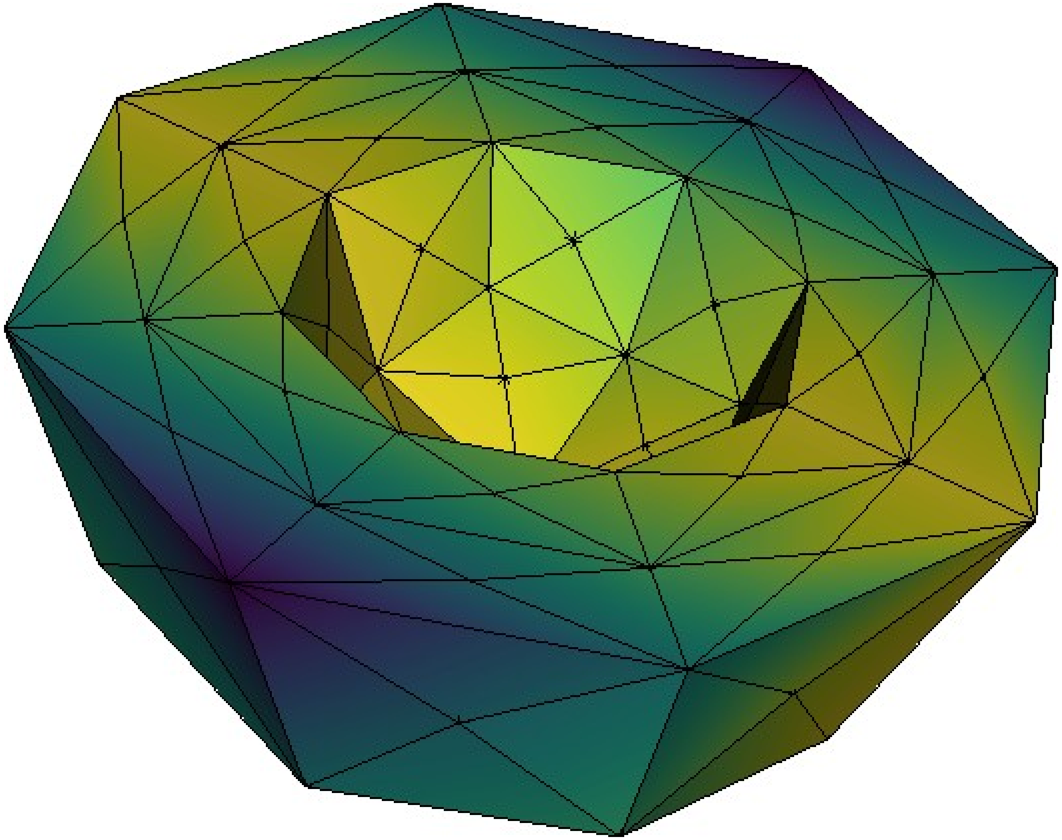
- Split the domain $\Omega \subset R^2$ of a function into "cells"
- Cells can be triangles or quadrilaterals
- Define a piecewise function: linear/polynomial on each cell, continuous at cell boundaries

In 3d:

- Split the domain $\Omega \subset R^3$ of a function into "cells"
- Cells can be tetrahedra, hexahedra, pyramids, prisms
- Define a piecewise function: linear/polynomial on each cell, continuous at cell boundaries

Piecewise polynomial approximation in 3d

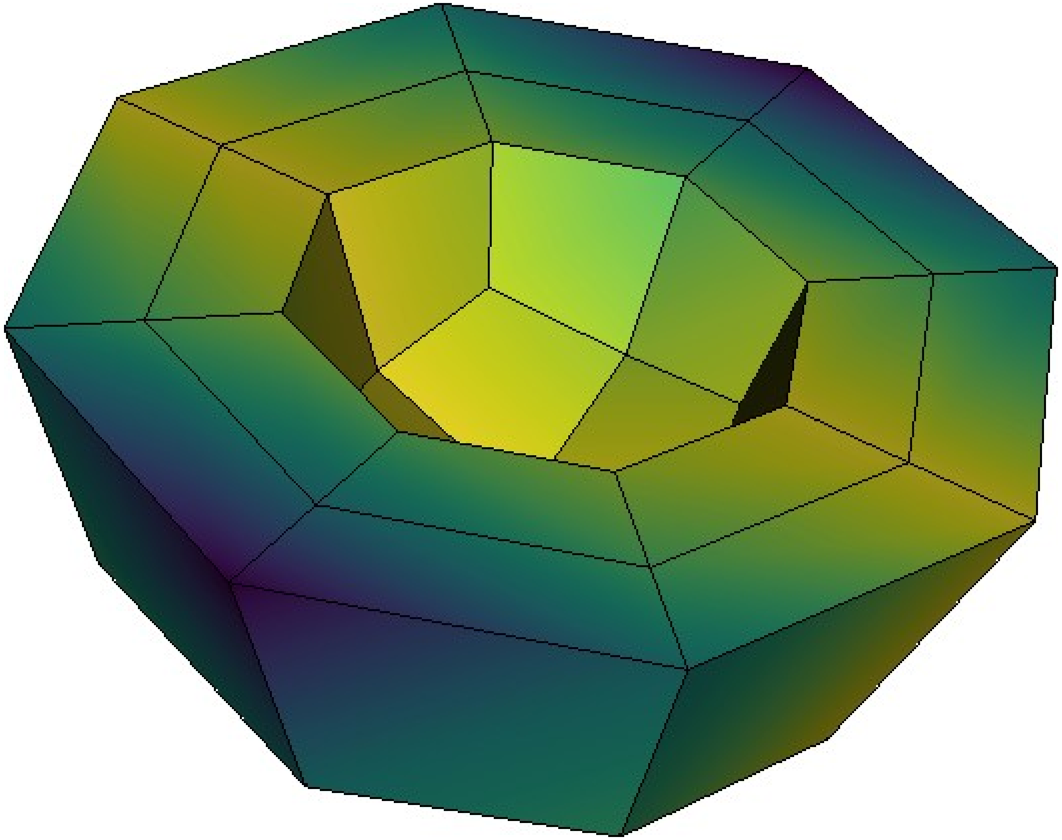
Example with tetrahedra:
Subdivide the domain into cells



We call the collection of cells the “mesh”.

Piecewise polynomial approximation in 3d

Example with hexahedra:
Subdivide the domain into cells



We call the collection of cells the “mesh”.

Piecewise polynomial approximation

"Higher order":

- The examples above used piecewise linear approximation
- Easily extendable to higher order:
  - piecewise quadratic
  - piecewise cubic
  - …
- On each interval/cell, the approximation is linear/quadratic/cubic/…

Notation: We say that the approximation is

- $P_1/P_2/...$: pw. linear/quadratic on triangles/tetrahedra
- $Q_1/Q_2/...$: pw. linear/quadratic on quads/hexahedra
- In 1d, $P_p=Q_p$

Piecewise polynomial approximation

Regardless of dimension and choice of cell:

Theorem:

- *f* is a function defined on a domain $\Omega \subset R^2$.
- $f_{h,p}$ is a piecewise function that interpolates *f*, defined on a mesh that subdivides Ω.
- h=the diameter of the largest cell of the mesh
- p=the polynomial degree used on the cells

Then:

$$\|f-f_{h,p}\| := \left(\int_\Omega |f(x)-f_{h,p}(x)|^2\right)^{1/2} \leq \frac{C_1(f,p,\Omega)}{p!} h^{p+1}$$

$$\|\nabla f-\nabla f_{h,p}\| := \left(\int_\Omega |\nabla f(x)-\nabla f_{h,p}(x)|^2\right)^{1/2} \leq \frac{C_2(f,p,\Omega)}{p!} h^{p}$$

Piecewise polynomial approximation

A note on terminology:

- We split the domain $\Omega\subset R^d$ of a function into "cells"
- In 1d: Cells are intervals
- In 2d: Cells can be triangles or quadrilaterals
- In 3d: Cells can be tetrahedra, hexahedra, pyramids, prisms

- The collection of cells is called the "mesh"

- We also use the term "triangulation"
  – even in 1d and 3d
  – even in 2d if we use quadrilaterals

Piecewise polynomial approximation

Summary:

- Subdivision of the domain into a mesh of cells + defining polynomial approximations on each cell generalizes the 1d construction

- As before, if we make cells small (*h* small), then the interpolation error decreases.

- Any function (within the class we care about) can be arbitrarily well approximated if we just put in enough computational work!

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