Finite element methods in scientific computing

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Lecture 3.91:

The ideas behind the finite element method

Part 2: Theory of (piecewise) polynomial approximation

Global polynomial approximation

Assume you have a function *f(x)* on an interval *[a,b]*.

Let us call its "interpolant" $f_p(x)$:

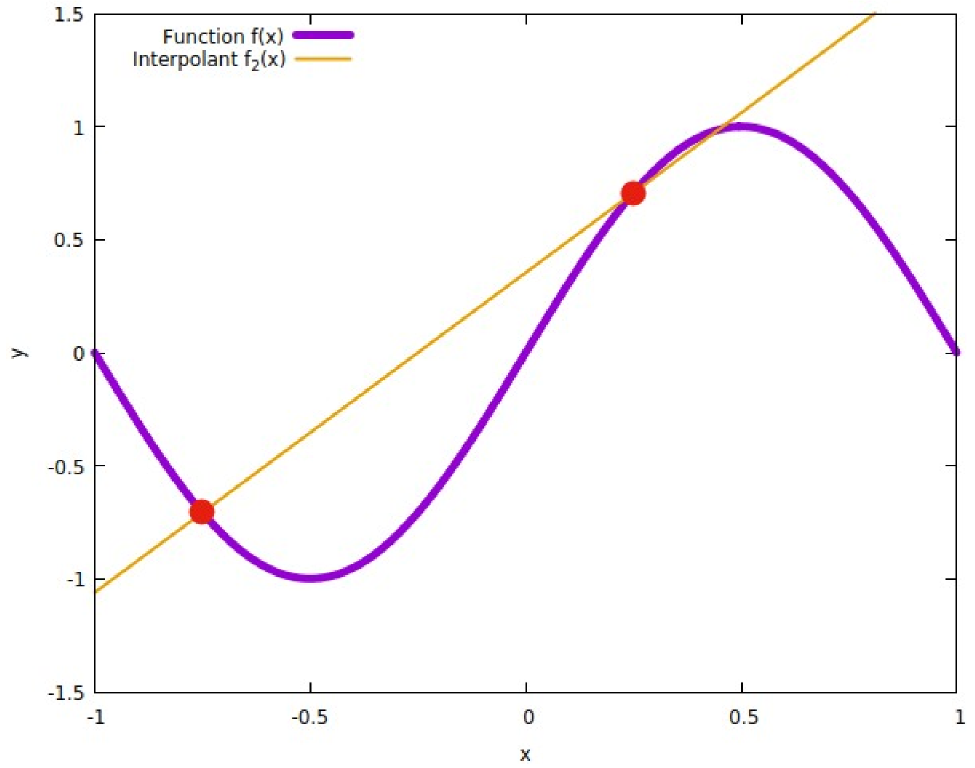
- Also a function on *[a,b]*
- Has polynomial degree *p*
- Is equal to *f(x)* at *(p+1)* points x_i:

$$f_p(x_i) = f(x_i) \qquad i = 1 \ldots p+1$$

Global polynomial approximation

Example for *f(x)=sin(πx)* on *[-1,1]*:

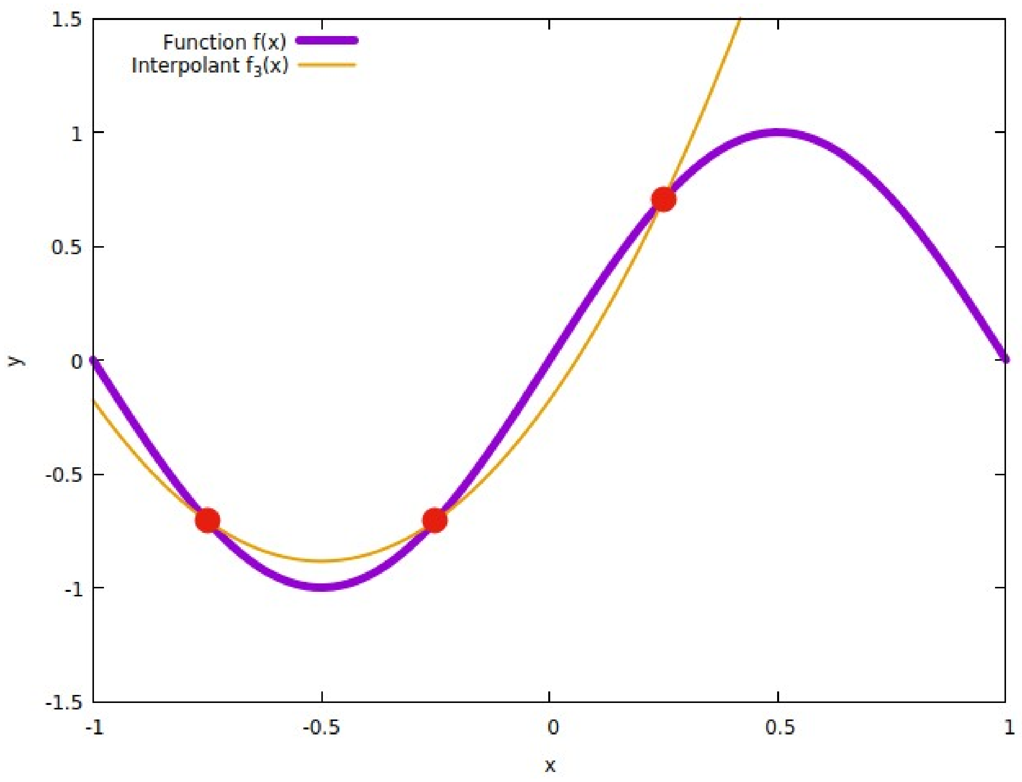
Choose $p=1$, $x_i=\{-0.75,+0.25\}$:



Global polynomial approximation

Example for *f(x)=sin(πx)* on *[-1,1]*:

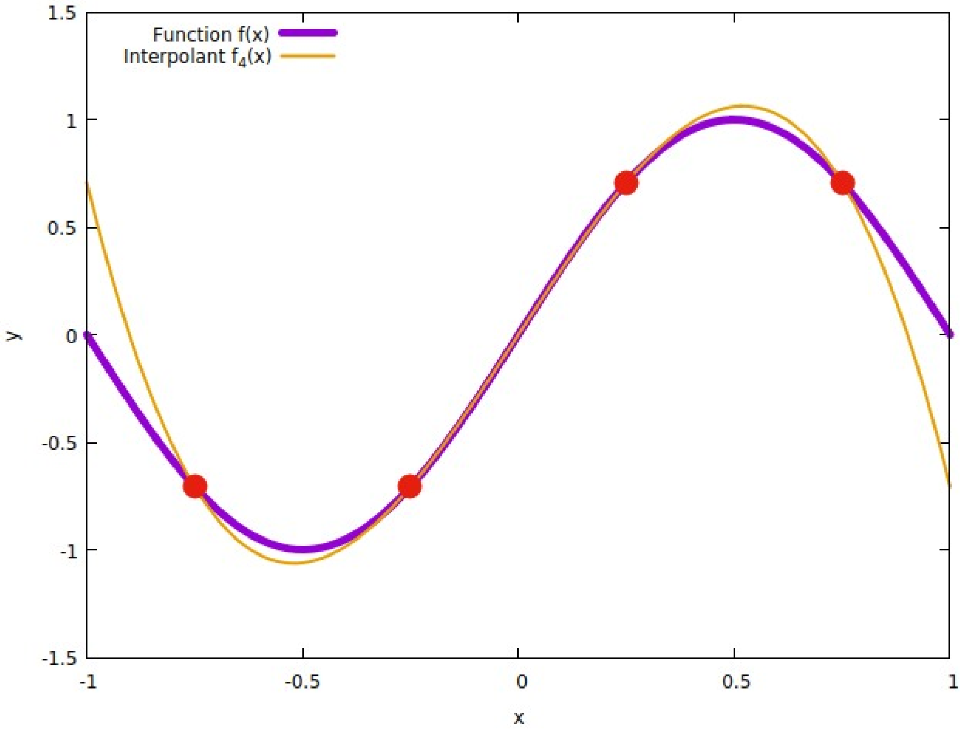
Choose $p=2$, $x_i=\{-0.75,-0.25,+0.25\}$:



Global polynomial approximation

Example for *f(x)=sin(πx)* on *[-1,1]*:

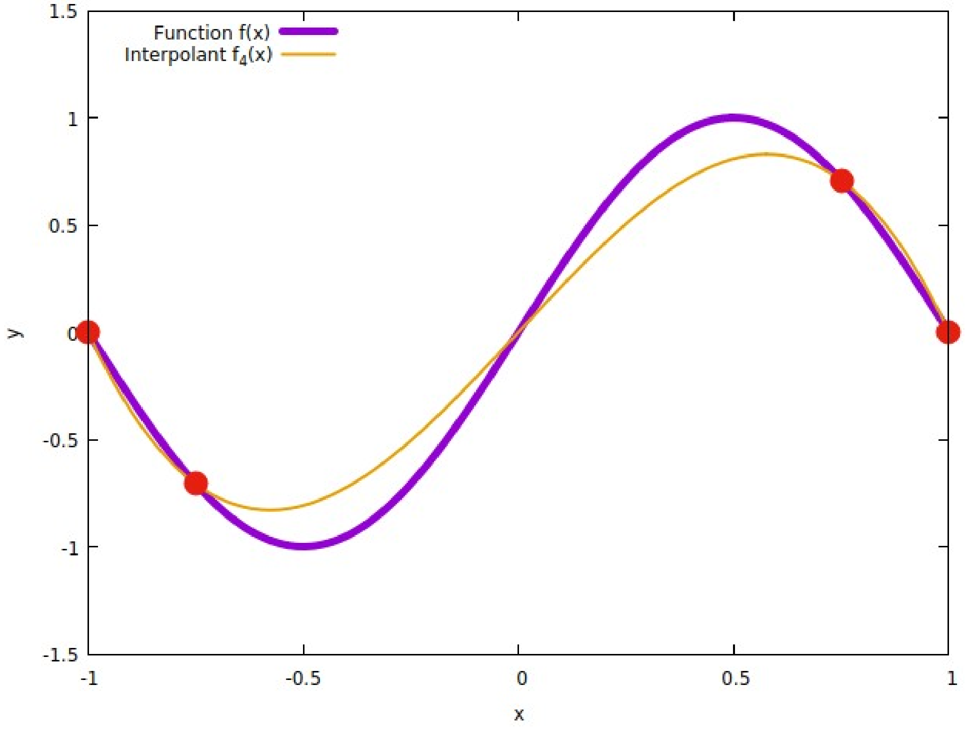
Choose $p=3$, $x_i=\{-0.75,-0.25,+0.25,+0.75\}$:



Global polynomial approximation

Example for *f(x)=sin(πx)* on *[-1,1]*:

Choose $p=3$, but different $x_i=\{-1,-0.75,+0.75,+1\}$:



Global polynomial approximation

Theorem (not optimal, but good enough):

If f is $p+1$ times continuously differentiable, then independent of the choice of the points x_i:

$$\max_{x\in[a,b]}|f(x)-f_p(x)| \le \frac{\max_{x\in[a,b]}|f^{(p+1)}(x)|}{p!}(b-a)^p$$

Global polynomial approximation

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$$\max_{x\in[a,b]}|f(x)-f_p(x)| \le \frac{\max_{x\in[a,b]}|f^{(p+1)}(x)|}{p!}(b-a)^p$$

Read this as follows:

$$\max_{x\in[a,b]}|f(x)-f_p(x)| \le C(f,p)\frac{(b-a)^p}{p!}$$

Global polynomial approximation

Theorem (not optimal, but good enough):

$$max_{x\in[a,b]}|f(x)-f_p(x)| \;\leq\; C(f,p)\frac{(b-a)^p}{p!}$$

Consequence:

- If *C(f,p)* does not grow too quickly, then

$$max_{x\in[a,b]}|f(x)-f_p(x)| \;\rightarrow\; 0 \qquad\qquad \text{as } p \text{ grows}$$

Problem: There are functions for which *C(f,p)* does grow rapidly.

Global polynomial approximation

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Example: $f(x)=1/x$ on [0.5, 1.5]:

$$\begin{aligned} C(f,p) &= max_{x\in[a,b]}|f^{(p+1)}(x)| \\ &= max_{x\in[\frac{1}{2},\frac{3}{2}]}|(-1)^{p+1}\ p!\ x^{-(p+2)}| \\ &= 2^{p+2}p! \end{aligned}$$

$$\begin{aligned} max_{x\in[a,b]}|f(x)-f_p(x)| &\leq \frac{C(f,p)}{p!}(b-a)^p \\ &= 2^{p+2}(b-a)^p = 2^{p+2} \end{aligned}$$

→ Polynomial approximant is not guaranteed to converge!

Global polynomial approximation

Theorem (not optimal, but good enough):

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Consequence:

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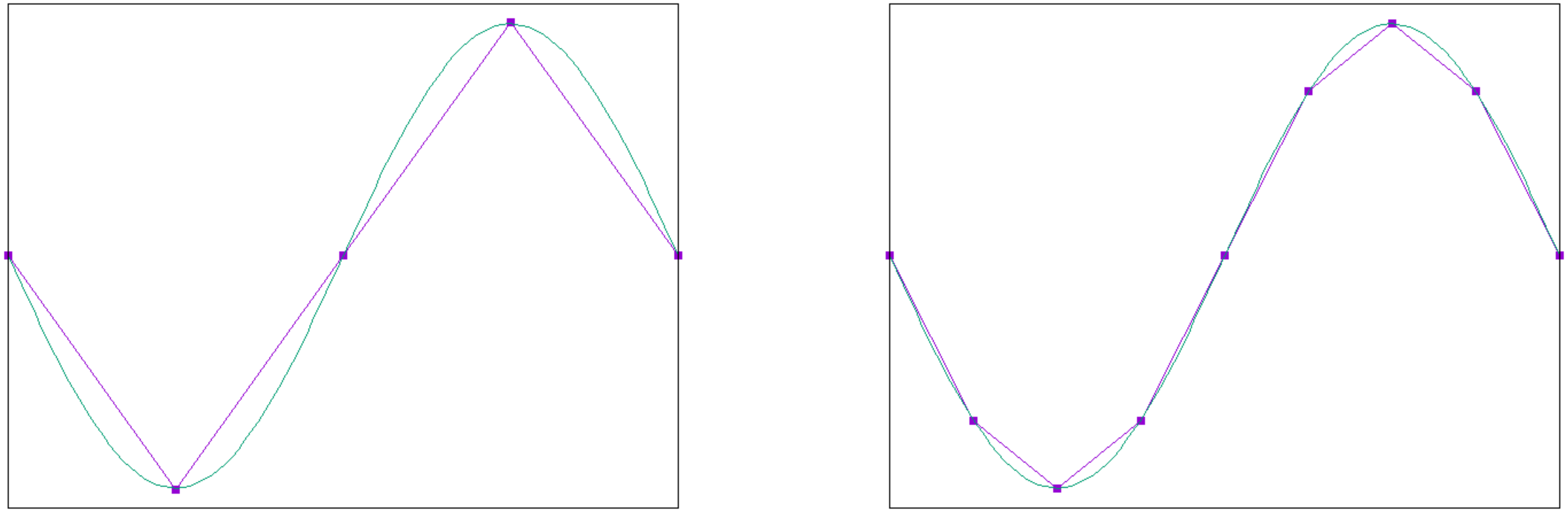
$$max_{x\in[a,b]}|f(x)-f_p(x)| \;\rightarrow\; 0 \qquad\qquad \text{as } p \text{ grows}$$

- **But:** Whether the “global interpolant” f_p converges to f depends on the function we try to approximate. This is undesirable.

Piecewise polynomial approximation

A better approach:

- Instead of increasing *p* on one interval
- ...keep *p* constant and instead split the interval into *n* pieces.



Green: The function $f(x)$ we want to approximate.

Purple: The piecewise linear approximant $f_h(x)$.

Piecewise polynomial approximation

A better approach:

- Instead of increasing *p* on one interval
- ...keep *p* constant and instead split the interval into *n* pieces.

Theorem:

$$\max_{x\in[a,b]}|f(x)-f_{h,p}(x)| \leq \frac{C(f,p)}{p!}\left(\frac{b-a}{n}\right)^p = \underbrace{\frac{C(f,p)(b-a)^p}{p!}}_{\text{constant}}\underbrace{\frac{1}{n^p}}_{\to 0}$$

Consequence: Pick *p*, choose enough intervals *n*, and you can make the difference as small as you want!

Piecewise polynomial approximation

Notation and more theory:

- We typically denote the diameter of intervals/cells by *h*
- Estimate will then look like this:

$$
\begin{aligned}
max_{x\in[a,b]}|f(x)-f_{h,p}(x)| &\leq \frac{C(f,p)}{p!}\left(\frac{b-a}{n}\right)^p \\
&= \underbrace{\frac{C(f,p)}{p!}}_{\text{constant}} h^p
\end{aligned}
$$

- For later purposes:

$$
\begin{aligned}
\|f-f_{h,p}\| &:= \left(\int_a^b |f(x)-f_{h,p}(x)|^2\right)^{1/2} &\leq \frac{C_1(f,p,a,b)}{p!} h^{p+1} \\
\|\nabla f-\nabla f_{h,p}\| &:= \left(\int_a^b |\nabla f(x)-\nabla f_{h,p}(x)|^2\right)^{1/2} &\leq \underbrace{\frac{C_2(f,p,a,b)}{p!}}_{\text{constant}} h^p
\end{aligned}
$$

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