MATH 676

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Finite element methods in scientific computing

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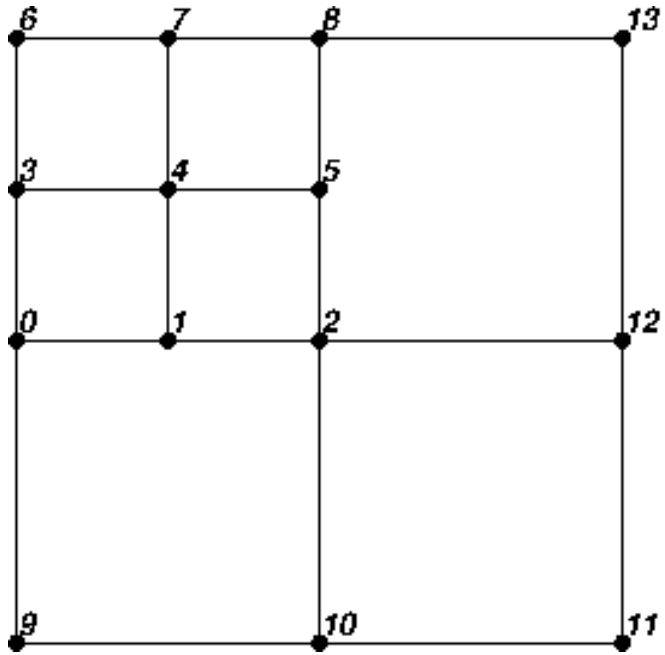
Lecture 16:

Hanging nodes and other constraints

Why constraints?

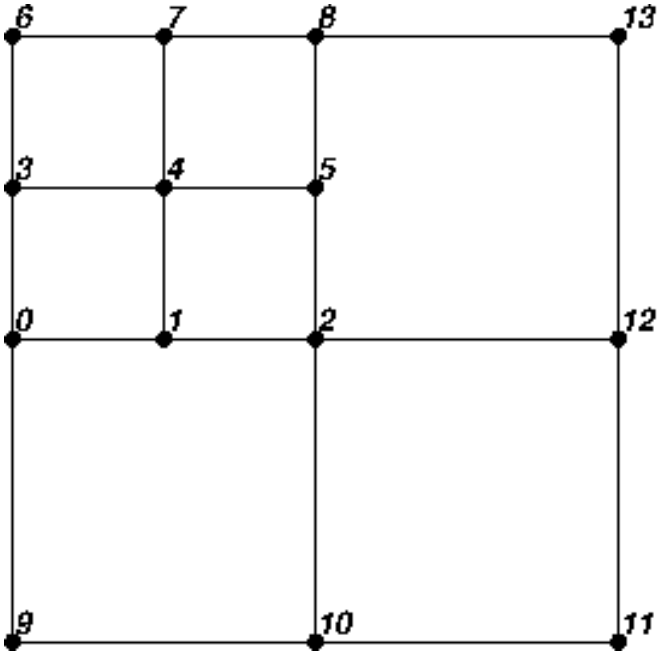
Consider this mesh, Q1 elements, and DoFs as enumerated:

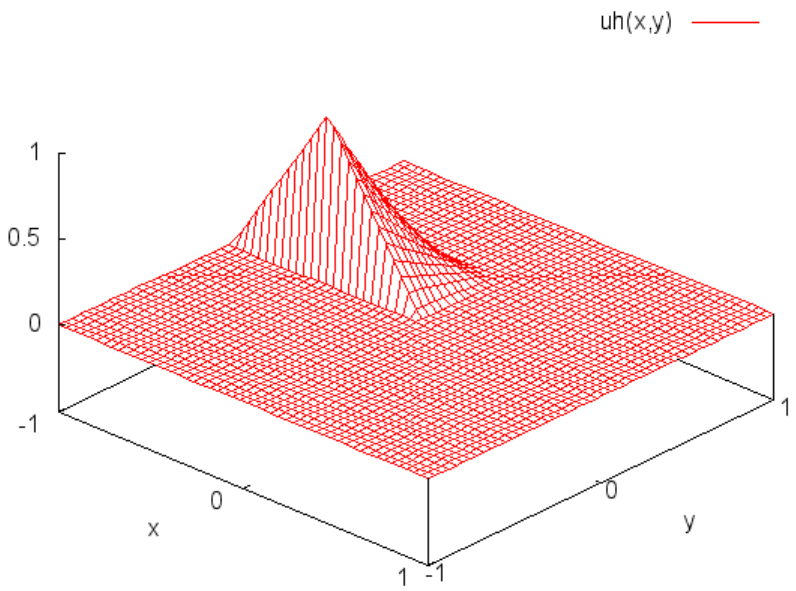


The corresponding space has dimension 14.

Why constraints?

Now consider a solution vector *U=(0,1,0,0,...)* and the function u_h associated with it:





Note: This function is not continuous!

Why constraints?

If our function space V_h has discontinuous functions:

- It is no longer a subspace of the usual H^1
- A bilinear form such as

$$a(u_h,\varphi_h) = \int_\Omega \nabla u_h \cdot \nabla \varphi_h \, dx$$

  no longer makes immediate sense

Resolution:

For spaces such as Q_1, we really need to *require* continuity!
We do so through *constraints*.

Why constraints?

Defining V_h via constraints:

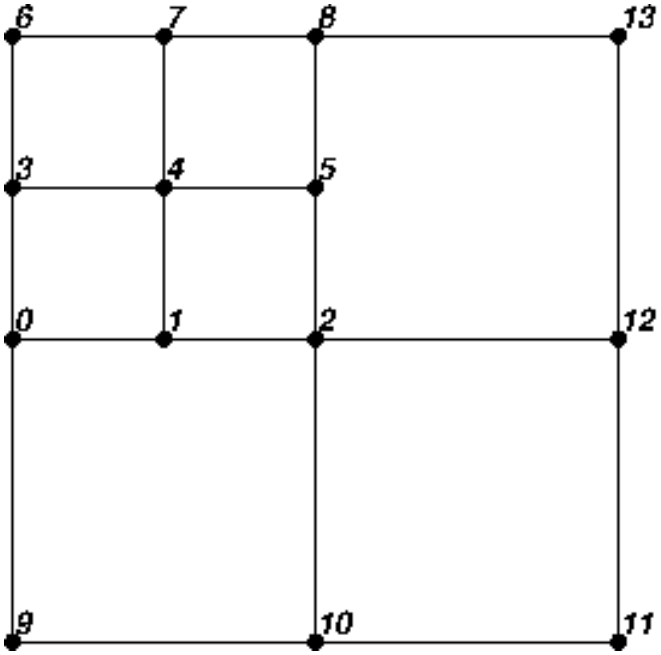
- Shape functions are defined on each cell as usual
- Functions in V_h are linear combinations of shape functions
- Functions in V_h are globally continuous

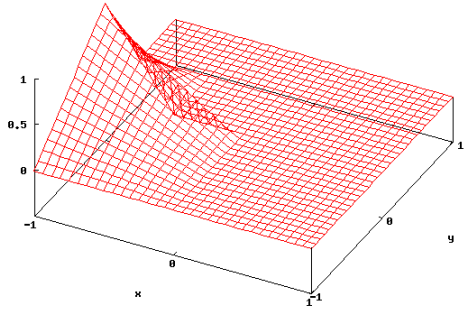
In other words:

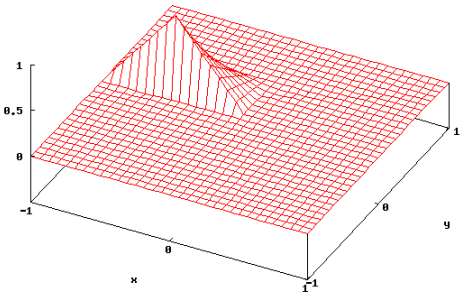
$$V_h = \left\{ v_h(x) = \sum_i V_i \varphi_i(x) : \ v_h(x) \text{ is continuous in } \Omega \right\}$$

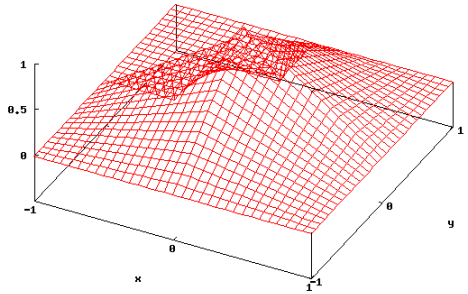
Why constraints?

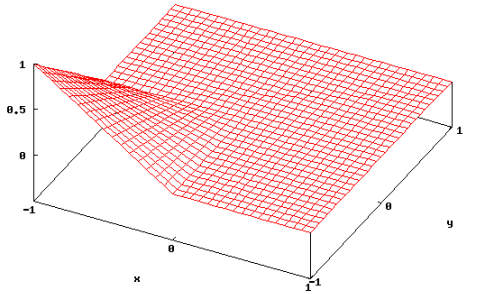
How do the shape functions look like:











Note: Not all of these functions are in V_h.

Which constraints?

Remember that we define V_h via constraints as:

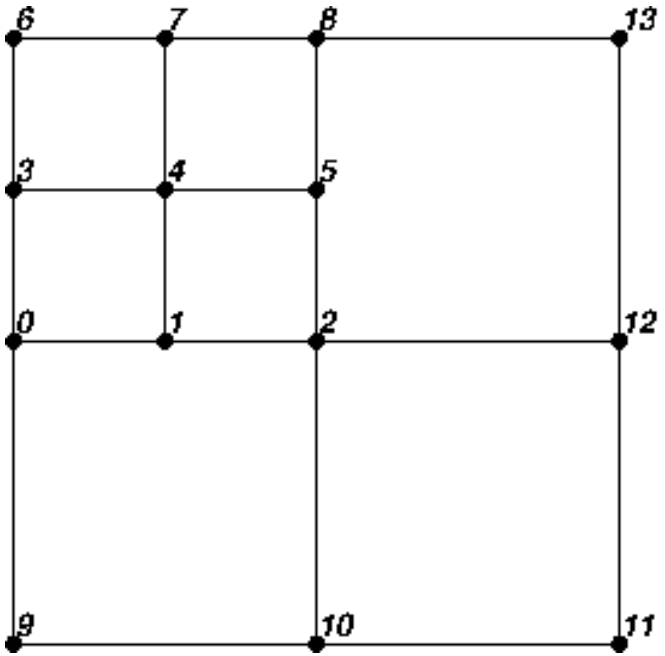
$$V_h = \left\{ v_h(x) = \sum_i V_i \varphi_i(x) : \; v_h(x) \text{ is continuous in } \Omega \right\}$$

The only possible discontinuities are along edges 0–1–2 and 2–5–8.

The function is in fact continuous if it is continuous at vertices 1 and 5!

That is:

$$V_1 = \frac{1}{2} V_0 + \frac{1}{2} V_2, \qquad V_5 = \frac{1}{2} V_2 + \frac{1}{2} V_8$$



Which constraints?

As a general rule:

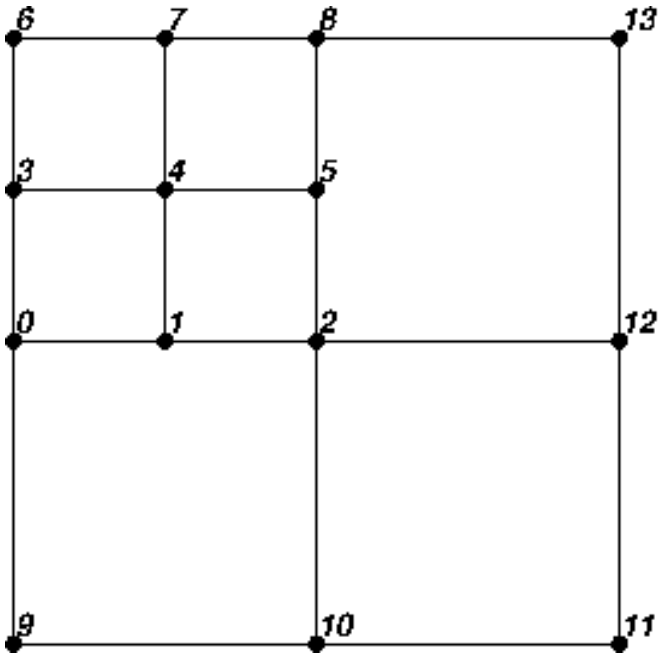
- When using hanging nodes, there is a subset *I* of *[0,n_dofs)* that is constrained
- These constraints have the form

$$V_i = \sum_{j=0}^{\text{n\_dofs}} \alpha_{ij} V_j \quad \forall i \in I$$

where most of the alphas are zero

- Here, for example:

$$V_1 = \frac{1}{2}V_0 + \frac{1}{2}V_2, \qquad V_5 = \frac{1}{2}V_2 + \frac{1}{2}V_8$$



- We can write this as

$$CV = 0, \qquad C \in \mathbb{R}^{\#\text{ constraints} \times \#\text{ dofs}}$$

Representation in deal.II

In deal.II:

- The constraints $CV=0$ for hanging nodes are represented by the deal.II class *ConstraintMatrix*
- *ConstraintMatrix* objects are built by the function *DoFTools::make_hanging_node_constraints*

Note: All of this works for *any* finite element, not just Q1. Furthermore, it also works for the *hp*-refinement case (see step-27).

Using constraints

Premise:

- The beauty of the FEM is that we do *exactly* the same thing on every cell
- Let us not destroy this property!
- That is: assembly on cells with hanging nodes should work exactly as on cells without.

Note: The mathematical and algorithmic details of dealing with constraints are complex (see Bangerth & Kayser-Herold, 2009). Therefore, let's discuss only the mechanics.

Using constraints

Define $\tilde{V}_h = \left\{ v_h(x) = \sum_i V_i \varphi_i(x) \right\}$

$$V_h = \left\{ v_h(x) = \sum_i V_i \varphi_i(x) : \ v_h(x) \text{ is continuous in } \Omega \right\}$$
$$= \left\{ v_h(x) = \sum_i V_i \varphi_i(x) : \ CV = 0 \right\}$$

Approach 1 (step-6):

- Step 1: Build matrix/rhs $\tilde{A}, \tilde{F}$ with all DoFs as if there were no hanging nodes.
- Step 2: Modify $\tilde{A}, \tilde{F}$ to get A, F out of this ("*condense*")
- Step 3: Solve $AU = F$
- Step 4: Get all components of *U* ("*distribute*")

Using constraints

Define $\tilde{V}_h = \left\{ v_h(x) = \sum_i V_i \varphi_i(x) \right\}$

$$V_h = \left\{ v_h(x) = \sum_i V_i \varphi_i(x) : \ v_h(x) \text{ is continuous in } \Omega \right\}$$
$$= \left\{ v_h(x) = \sum_i V_i \varphi_i(x) : \ CV = 0 \right\}$$

Approach 2 (step-22):

- Step 1: Build *local* matrix/rhs $\tilde{A}^K, \tilde{F}^K$ with all DoFs as if there were no hanging nodes.
- Step 2: Modify when copying local contributions into global matrices A, F ("*copy_local_to_global*")
- Step 3: Solve $AU = F$
- Step 4: Get all components of *U* ("*distribute*")

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