MATH 651: Numerical Analysis II

Instructor: Prof. Wolfgang Bangerth
bangerth@colostate.edu

Homework assignment 2 – due 9/27/2019

Problem 1 (Stability of Lagrange interpolation). Let's say you want to approximate the function $f(x)$ with a single, global, high order polynomial p_{N-1}. To this end, assume that we evaluate f at a set of points $\{x_i\}_{i=1}^N$ to obtain $y_i = f(x_i)$.

We will again assume that f is expensive to evaluate. The goal of this exercise is to assess what happens if we can only evaluate f *up to some measurement error*. In other words, we don't know the exact y_i values, but only some $\tilde{y}_i \approx y_i$, and if we interpolate these values, we will also only obtain an interpolation function $\tilde{p}_{N-1}$ that we hope is still close to p_{N-1}.

Mathematically, any good method should ensure that if $\|\mathbf{y} - \tilde{\mathbf{y}}\|$ is small, then $\|p_{N-1} - \tilde{p}_{N-1}\|$ is also small in some norm (e.g., the infinity norm) – or more specifically, we want that

$$\|p_{N-1} - \tilde{p}_{N-1}\|_\infty \leq C\|\mathbf{y} - \tilde{\mathbf{y}}\|_\infty$$

for some constant C of *moderate* size for any perturbation $\tilde{\mathbf{y}}$ or $\mathbf{y}$. (The norm on the left is the infinity norm on functions, i.e., the maximal absolute value within the domain of the function; the norm on the right is the maximum norm on vectors, i.e., the maximal absolute value over all entries of the vector.) If such a constant C exists, and if it is of "reasonable" size, then we say a method is "stable" in the presence of noise.

To assess this, let us assume that $f(x) = 0$. Then $y_i = 0$ and consequently we know that $p_{N-1}(x) = 0$. Furthermore, if we choose $\tilde{y}_i$ randomly between -0.001 and 0.001, then $\|\mathbf{y} - \tilde{\mathbf{y}}\|_\infty = \|\tilde{\mathbf{y}}\|_\infty \leq 0.001$. We can then estimate C as

$$C \geq \frac{\|p_{N-1} - \tilde{p}_{N-1}\|_\infty}{\|\mathbf{y} - \tilde{\mathbf{y}}\|_\infty} = \frac{\|\tilde{p}_{N-1}\|_\infty}{0.001}.$$

The computable term on the right hand side is only a lower bound for C because we consider only a single realization of the random perturbation of the data.

(a) Using a set of N *equidistant* points x_i with $x_1 = -1$ and $x_N = 1$, write a function that evaluates the interpolating polynomial $\tilde{p}_{N-1}$ for these x_i and randomly chosen $\tilde{y}_i$ as specified above. Plot $\tilde{p}_{N-1}(x)$ and visually evaluate the size of the error due to the inexact measurements, $\max_{-1\leq x\leq 1} |p_{N-1}(x) - \tilde{p}_{N-1}(x)| = \max_{-1\leq x\leq 1} |\tilde{p}_{N-1}(x)|$ for a few different values of N. Plot the maximal error as a function of N for a number of values of N in the range $1 \ldots 30$. Describe the qualitative behavior you observe for the lower bound of C discussed above as N becomes larger.

(b) Repeat the experiment but with the interpolation points chosen as the roots of the Chebyshev polynomial (if necessary shifted to the interval $[-1, 1]$).

(c) What conclusions can you draw about the suitability of higher order polynomial interpolation in the presence of inaccurate data? How would *piecewise linear* interpolation fare in this context?

(30 points)

Problem 2 (Adaptive interpolation). Consider approximating the function $f(x) = \sin\frac{1}{x}$ on the interval $[0.05, 1]$ with a piecewise linear function. If you plot this function on the interval, you will realize that this is an example of a function that is nice and smooth in parts of its domain but not in other regions.

(a) Write an interpolation procedure that starts with a single interval and iteratively replaces *every* interval by its two halves.

For each step in this iteration, evaluate an approximation of the maximal error between the original function f and your current piecewise linear approximation. Plot this error as a function of the number of evaluation points you use for your current approximation.

(b) Write an adaptive interpolation procedure that starts with a single interval and iteratively finds the interval that has the worst approximation (using your criterion of choice) and replaces it by its two halves.

For each step in this iteration, evaluate an approximation of the maximal error between the original function f and your current piecewise linear approximation. Plot this error as a function of the number of evaluation points you use for your current approximation.

For both methods, show the interpolant p^h you obtain with $N = 129$ evaluation points, along with the location of these points.

Compare the two approaches. Which requires fewer function evaluations for a given error level (say, to achieve an approximation quality of $\|f - p^h\|_\infty \leq 10^{-6}$)? **(30 points)**

Problem 3 (Spline interpolation). Spline interpolation is typically used in cases where (i) $f(x)$ is expensive to evaluate, (ii) one would need $f'(x)$ for some purpose, but this is not available. In that case, one would try to approximate f by a piecewise polynomial p^h in such a way that $\frac{d}{dx}p^h$ can be evaluated everywhere – i.e., p^h must be continuously differentiable.

One example is trying to find the root of a function $f(x)$ (i.e., that x^* for which $f(x^*) = 0$) using Newton's iteration that computes $x_{k+1} = x_k - \frac{f(x_k}{f'(x_k)}$.

(a) Consider $f(x) = xe^x - 1$. Use Newton's method to compute its root x^* to at least six digits of accuracy, starting with $x_0 = 0.5$. (In other words, I'm asking you to compute $W(1)$ where W is *Lambert's W function*.)

(b) Write a procedure that computes a cubic spline approximation p^h of $f(x)$ using N equally spaced points x_i on the interval $[0, 1]$. Compute the root $x^{N,*}$ of p^h to at least six digits of accuracy, starting with $x_0 = 0.5$. Demonstrate numerically that $x^{N,*} \to x^*$ as $N \to \infty$ and plot the error $|x^{N,*} - x^*|$ as a function of N for $N = 2 \ldots 100$.

(40 points)