MATH 451: Introduction to Numerical Analysis II

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Homework assignment 2 – due Friday 2/16/2018

Problem 1 (Reduction of a real, symmetric matrix to tridiagonal form). Take the following sequence of matrices:

$$A_1 = \frac{1}{1}\begin{pmatrix}2\end{pmatrix}, \qquad A_2 = \frac{1}{2}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}, \qquad A_3 = \frac{1}{3}\begin{pmatrix}4 & 1 & 1\\ 1 & 4 & 1\\ 1 & 1 & 4\end{pmatrix}, \qquad A_4 = \frac{1}{4}\begin{pmatrix}5 & 1 & 1 & 1\\ 1 & 5 & 1 & 1\\ 1 & 1 & 5 & 1\\ 1 & 1 & 1 & 5\end{pmatrix}, \qquad \ldots$$

The general form of these matrices is that $(A_n)_{ij} = \frac{1}{n}, (A_n)_{ii} = \frac{n+1}{n}$.

Implement an algorithm (in a programming language or system of your choice) that performs a sequence of Householder reflections (i.e., computes a sequence of products of the form $(I - 2ww^T)A(I - 2ww^T)$ with appropriately chosen vectors w) to reduce A_n to tridiagonal form.

Demonstrate that your algorithm works correctly by showing the reduced forms for each of the matrices $A_3, \ldots, A_8$ and verifying that it is indeed symmetric and tridiagonal. **(25 points)**

Problem 2 (QR decomposition). Take the following sequence of matrices:

$$B_1 = \begin{pmatrix}2\end{pmatrix}, \qquad B_2 = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}, \qquad B_3 = \begin{pmatrix}2 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}, \qquad B_4 = \begin{pmatrix}2 & 1 & 0 & 0\\ 1 & 2 & 1 & 0\\ 0 & 1 & 2 & 1\\ 0 & 0 & 1 & 2\end{pmatrix}, \qquad \ldots$$

The general form of these matrices is that $(B_n)_{ii} = 2, (B_n)_{i,i+1} = 1, (B_n)_{i,i-1} = 1$, with all other entries zero. These matrices are all symmetric and tridiagonal.

Implement an algorithm that computes the QR decomposition of a real, symmetric, tridiagonal matrix A into an orthogonal matrix Q and a triangular matrix R, using a sequence of rotation matrices.

Demonstrate that your algorithm works correctly by showing that for each of the matrices $B_2, \ldots, B_8$, you have that $Q^T Q = I$, R is indeed triangular, and the product of the matrices QR indeed results again in the matrix B_n you started from. **(25 points)**

Problem 3 (Repeated QR decomposition). Take a matrix $B^{(1)} = B$ from the previous problem and compute its QR decomposition $B^{(1)} = Q^{(1)}R^{(1)}$. Then set $B^{(2)} = R^{(1)}Q^{(1)}$, and repeat the procedure.

Show experimentally for each of $B_2, \ldots, B_8$ that if you keep doing this, the matrices $B^{(k)}$ converge to a diagonal matrix. Use this to estimate the eigenvalues of the matrices B_n.

Verify that your algorithm is correct by comparing your eigenvalue estimates with the exact eigenvalues computed through a separate approach (for example, using Matlab's `eig` command, Maple's `eigenvalues` command, etc). **(25 points)**

Problem 4 (Putting it all together). Put it all together by starting with the matrices from Problem 1, reducing these matrices to tridiagonal form, and the doing the iterated QR decomposition to compute approximations of the eigenvalues.

You may want to compare with the exact eigenvalues of A_n – which are equal to 2 with multiplicity one, and 1 with multiplicity $n - 1$. **(25 points)**