MATH 620: Variational Methods and Optimization I

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| Lectures: | Engineering E 206, Mondays/Wednesdays/Fridays, 12-12:50pm |
| Office hours: | Wednesdays, 1-2pm; or by appointment. |

Homework assignment 3 – due Friday 10/12/2018

Problem 1 (Euler-Lagrange equations). We have proven rigorously that any sufficiently smooth minimizer (say, if the minimizer happens to be in C^2) of the functional

$$I(u) = \int_a^b f(x, u(x), u'(x)) \, \mathrm{d}x$$

has to satisfy the Euler-Lagrange equations

$$f_u(x, u(x), u'(x)) = \frac{\mathrm{d}}{\mathrm{d}x} f_\xi(x, u(x), u'(x))$$

for all $x \in (a, b)$.

The energy related to the vertical deflection of a thin (one-dimensional) beam subject to a vertical gravity field with strength g is given by

$$I(u) = \int_a^b \left[\mu \left(\sqrt{1 + u'(x)^2} - 1 \right) - gu(x) \right] \, \mathrm{d}x,$$

where μ is related to the elasticity constants of the material.

(a) Derive the Euler-Lagrange equations (including boundary conditions) for $u(x)$ for this problem if you try to minimize the energy over the set of functions

$$D := \left\{ \varphi \in C^2 : \varphi(a) = 1, \varphi(b) = 1 \right\}.$$

(b) Can you prove that the solution $\bar{u}$ of the (Euler-Lagrange) differential equations is the unique minimizer of $I(u)$?

(25 points)

Problem 2 (Euler-Lagrange equations). In general, the Euler-Lagrange equations will be a (system of) nonlinear ordinary differential equation. Most often, they will not be exactly solvable. But occasionally, we can solve simplified problems.

If you take Problem 1, consider the case of stiff materials that do not deform very much. In that case, u' will be small, and we can use the approximation

$$\sqrt{1 + y} \approx 1 + \frac{1}{2}y.$$

(This is just Taylor expansion around $y = 0$.) Use this to define an approximate energy functional $\tilde{I}(u)$.

(a) Derive the Euler-Lagrange equations (including boundary conditions) for $u(x)$ for this approximate problem with the same D as before.

(b) Can you prove that the solution $\bar{u}$ of the (Euler-Lagrange) differential equations is the unique minimizer of $I(u)$?

(c) Actually solve this problem, i.e., find $\bar{u}$ that satisfies the Euler-Lagrange equations.

(20 points)

Problem 3 (The brachystochrone problem). We have started the semester by considering Newton's minimal resistance problem and Bernoulli's brachystochrone problem (the "bead on a wire"). The former is a bit more complicated because the right integration bound b depends on the solution. But for the former, the case is easy: We have

$$I(u) = \int_0^L \sqrt{\frac{\frac{1}{2}(1 + u'(x)^2)}{gH - gu(x)}}\,\mathrm{d}x$$

and

$$D := \left\{\varphi \in C^2 : \varphi(0) = H, \varphi(b) = 0.\right\}$$

State the Euler-Lagrange equations and boundary conditions any sufficient smooth minimizer $\bar{u}$ would have to satisfy. Is the solution unique? **(15 points)**

If you can, also solve the Euler-Lagrange equations. The solution can of course be found on the internet or in any number of books, but if you want to get these bonus points, you will need to show step by step how you solve the equations – this is going to be non-trivial. **(10 bonus points)**

Problem 4 (Multiple dimensions). Everything we have done in class was based on functions $u(x)$ of a single argument $x \in [a, b] \subset \mathbb{R}$. But in reality, it is not very difficult to derive the same kind of Euler-Lagrange equations also for higher dimensional functions $u(\mathbf{x})$, $\mathbf{x} \in \Omega \subset \mathbb{R}^n$.

To this end, let us assume that we want to find a minimizer of

$$I(u) = \int_\Omega f(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x}))\,\mathrm{d}^n x$$

and

$$D := \left\{\varphi \in C^2(\Omega) : \varphi(\mathbf{x}) = g(\mathbf{x})\ \forall \mathbf{x} \in \partial\Omega\right\}.$$

Go through the one-dimensional derivation of the Euler-Lagrange equations and adapt it as appropriate to derive the (now partial) differential equation any sufficiently smooth minimizer $\bar{u} \in D$ has to satisfy. State both the equations and boundary conditions. **(25 points)**

Problem 5 (Multiple dimensions). The generalization to higher dimensions of the first problem is to look for the minimizer of the functional

$$I(u) = \int_\Omega \left[\mu\left(\sqrt{1 + |\nabla u(\mathbf{x})|^2} - 1\right) - gu(\mathbf{x})\right]\,\mathrm{d}^n x,$$

State the Euler-Lagrange equations for this problem.

Next, apply the same simplification we considered in Problem 2 and again derive the corresponding Euler-Lagrange equations.

(15 points)

Bonus problem (Subdifferentials). In class and in the previous homework, we found that the subdifferentials of the functions $f_1(\mathbf{x}) = \|\mathbf{x}\|_1$, $f_2(\mathbf{x}) = \|\mathbf{x}\|_2$, and $f_\infty(\mathbf{x}) = \|\mathbf{x}\|_\infty$ at the origin, i.e., $\partial f(0)$, are exactly the unit balls defined by the l_∞, l_2, and l_1 norms. This suggests the conjecture that the subdifferential $\partial f_p(0)$ of $f_p(\mathbf{x}) = \|\mathbf{x}\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$ might be the unit ball $\{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\|_q \leq 1\}$, where q depends on p so that they satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Prove or disprove this. **(10 bonus points)**