MATH 545: Partial Differential Equations I

Instructor: Prof. Wolfgang Bangerth
Weber 214
bangerth@colostate.edu

Lectures: Engineering E 206, Mondays/Wednesdays/Fridays, 10-10:50am
Office hours: Wednesdays, 1-2pm; or by appointment.

Homework assignment 3 – due Friday 10/12/2018

Problem 1 (Well-posedness vs. ill-posedness). When you reach for an object, your hand will almost certainly be able to touch it to within a couple of millimeters accuracy. That is because your brain is really good at estimating how far away something is based on what information it gets from the eyes. How does your brain do that? In essence, it is able to tell that an object appears at a different angle against the "straight-ahead direction" in one eye than in the other. (This difference is called parallax.)

Let us verify that this problem is well-posed. To this end, measure the distance between your eyes in meters (let's call that distance E) and assume that an object is straight ahead at a distance $L = 0.5$m. Your eyes and the object then form an isosceles triangle. Compute the angles α and β of this triangle at the two eyes.

Now let's reverse this. Write down the formula that computes the *height* L of a triangle above the base line given angles α, β, and E. Then do the following experiments using the values you have for α, β, E:

(a) Verify that you get back 0.5m if you plug in the values you have.

(b) Perturb the angles α, β by small amounts, say by $\delta = 0.05° = 0.05\frac{2\pi}{360}$. (That seems like a reasonably small uncertainty in your brain's ability to determine angles if one puts it in context: The diameter of the full moon is about 0.5°.) Create five sets of $\tilde{\alpha}, \tilde{\beta}$ values that are perturbed from the exact values α, β by about this uncertainty and for each compute what distance $\tilde{L}$ the eye would estimate that the object is away from you.

(c) State whether you think that the problem of finding L from α, β, E is well-posed for distances in the range of about half a meter.

What about objects that are far away? Say, the distance to a Colorado mountain 10km away? Let us repeat this whole experiment but with $L = 10^3$m:

(d) Choose the same E as before and an object $L = 10^3$m straight ahead. Compute the corresponding angles α, β. Then verify that if you plug these values for α, β, E into your formula, you get back 10^3m.

(e) Perturb the angles α, β by small amounts as above. Create five sets of $\tilde{\alpha}, \tilde{\beta}$ values that are perturbed from the exact values α, β by about this uncertainty and for each compute what distance $\tilde{L}$ the eye would estimate that the object is away from you.

(f) State whether you think that the problem of finding L from α, β, E is well-posed for distances in the range of several kilometers.

(You will find out that estimating distances by parallax does not work well for large distances. So how does the brain do it? It takes into account that (i) air is generally hazy, and consequently objects that look less colorful or sharp are further away, (ii) objects that are farther *appear* smaller, and if you know how large something is – say, a tree – then you can use this to estimate how far away it is.

Both of these effects work well in conditions we are familiar with. But there are situations where we find ourselves unable to determine distances. One is if you are in a plane: It is really hard for our brains to estimate how high we are because we are not familiar with the shape and size of objects when viewed from above. Another is at high altitude: When you look from one mountain at another, it will often look impossibly steep because we cannot assess distances: the air is so clear that estimating based on haze does not work; and a large boulder will look essentially the same as a medium-sized rock, so no estimating based on comparing physical and apparent size either. Because being out and about in the mountains is beautiful, I encourage you go check out this effect yourself!) **(20 points)**

Problem 2 (Fourier series). The Fourier series on $[-L, L]$ of a function $f(x)$ that is piecewise smooth is given by

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

where

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx,$$

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \qquad B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

Calculate the Fourier series on $[-\pi, \pi]$ of the function

$$f(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ -1 & \text{for } x < 0. \end{cases}$$

(10 points)

Problem 3 (Gibbs phenomenon). Consider the same function $f(x)$ and its Fourier series as in the previous problem. Using a computer graphing program such as Maple, Matlab, or Mathematica (or whatever else you deem fit for the task), show graphs for the range $x \in [-\pi, \pi]$ of the following:

- $f(x)$ and the first 3 terms of its Fourier series;
- $f(x)$ and the first 6 terms of its Fourier series;
- $f(x)$ and the first 15 terms of its Fourier series;
- $f(x)$ and the first 30 terms of its Fourier series.

You will see that the plots of the first terms of the Fourier series approximate $f(x)$ increasingly well, but that there are over- and undershoots around the location where $f(x)$ has a jump (i.e. at $x = 0$). These oscillations are called *Gibbs phenomenon*.

Conjecture what the Fourier series converges to for points $x < 0$, $x = 0$, and $x > 0$. **(10 points)**

Problem 4 (Periodic continuation). Repeat the plots you already generated for Problem 2, but this time show $f(x)$ and the first 3, 6, 15, and 30 terms of its Fourier series in the interval $[-3\pi, 3\pi]$. Interpret what you see.

(5 points)

Problem 5 (Solutions of the Laplace equation). In class we have seen that one can solve the Laplace equation on a rectangle

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

by chopping the solution up into four parts $u = u_1 + u_2 + u_3 + u_4$, where each of the u_i satisfies boundary conditions on one part of the boundary and is zero on the other three parts. We have constructed u_4 in class that satisfies the boundary conditions at $x = 0$. Construct u_2 that satisfies the boundary conditions at the right edge, $x = L$, and is zero on all other parts of the boundary. Identify the steps that differ from what we did in computing u_4. **(20 points)**

Problem 6 (Uniqueness of solutions of the Laplace equation). In class, we discussed that the heat equation has solutions (existence) by explicitly constructing them, and that there is only one solution (uniqueness).

We showed the latter by assuming that there are two solutions u_1, u_2 that satisfy the PDE, boundary conditions, and initial conditions. We then introduced the difference $\delta = u_1 - u_2$ and stated the equations that δ has to satisfy: the homogenous PDE as well as homogenous boundary and initial conditions. If you go back to your notes, you will see that it wasn't particularly hard to derive that $\delta = 0$ by multiplying the PDE for δ, integrating over time and space, and integrating by parts where necessary to arrive at an equation that had two non-negative terms that added up to zero, and therefore proved that each term had to be zero. Thus we have $\delta = 0$ which implies $u_1 = u_2$, which in non-mathematical language reads that every two solutions of the heat equation are equal, or in other words that there is exactly one unique solution.

For the Laplace equation we have shown existence of solutions in class. Prove uniqueness by following the same recipe: assume that there are two solutions $u_1(x, y), u_2(x, y)$ of

$$\begin{aligned} -\Delta u &= q && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

where $q = q(x, y)$ and Ω is a bounded domain in $\mathbb{R}^2$ (not necessarily a rectangle). Then introduce the difference $\delta = u_1 - u_2$, derive the equation and boundary conditions δ has to satisfy, then multiply the PDE for δ by δ itself, integrate, and integrate by parts. Show that this implies that $\delta = 0$, i.e. $u_1 = u_2$. Does this result also hold for domains $\Omega \subset \mathbb{R}^3$? How about even higher dimensions? **(20 points)**

Problem 7 (Laplacian in a polar coordinate system.) We can identify each point in the plane by specifying its x and y coordinates. Alternatively, we can state its distance r from the origin as well as the angle θ the vector from the origin to the point forms with the positive x-axis. These two coordinate systems are related by

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan \frac{y}{x}.$$

If we are given a function $F(r, \theta) = F(r(x, y), \theta(x, y))$, we can compute derivatives with respect to each set of coordinates using the chain rule:

$$\frac{\partial F(r(x, y), \theta(x, y))}{\partial x} = \frac{\partial F(r, \theta)}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial F(r, \theta)}{\partial \theta} \frac{\partial \theta}{\partial x},$$

and similarly for $\frac{\partial}{\partial y} F(r(x, y), \theta(x, y))$.

Show that the following identity holds:

$$\Delta F(r(x, y), \theta(x, y)) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} F(r, \theta) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} F(r, \theta).$$

Here,

$$\Delta F(r(x, y), \theta(x, y)) = \frac{\partial^2 F(r(x, y), \theta(x, y))}{\partial x^2} + \frac{\partial^2 F(r(x, y), \theta(x, y))}{\partial y^2}.$$

The calculations that lead to this result are a bit tedious and lengthy, but since you know the answer you should be able to find a way by simply applying the chain rule often enough and suitably simplifying terms. **(15 points)**