MATH 651: Numerical Analysis II

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Homework assignment 1 – due 9/12/2017

Problem 1 (Lagrange interpolation).

(a) Compute the Lagrange interpolation polynomials $L_{3,k}, k = 1 \dots 4$, for the points $x_1 = 1$, $x_2 = 2$, $x_3 = 1.5$ and $x_4 = 1.6$.

(b) Calculate the interpolating polynomial for the data set where $y_k = \log x_k$ at the four points x_k. Write the polynomial in the form $p_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. **(10 points)**

Problem 2 (Lagrange interpolation error). The polynomial $p_3(x)$ calculated in Problem 1 by construction interpolates the function $f(x) = \log x$. Based on the theorem discussed in class, compute an upper bound for the error on the interval $[1, 2]$, using the theorem that states how large $|f(x) - p_3(x)|$ can at most be. Compare the accuracy of this bound visually by plotting both functions as well as their difference. **(15 points)**

Problem 3 (Lagrange interpolation error). We have discussed in class that the interpolation error theorem yields an exponential growth of the error between $f(x)$ and its interpolant $p_{N+1}(x)$ as $N \to \infty$ for the function $f(x) = \frac{1}{x}$ if the interpolation points are located in the interval $x_i \in \left[\frac{1}{2}, \frac{3}{2}\right]$.

However, the theorem only yields an *upper bound*. It does not imply that the difference

$$\max_{\frac{1}{2} \le x \le \frac{3}{2}} |f(x) - p_{N-1}(x)|$$

really has this behavior. Let us assess this in practice:

(a) Using a set of N *equidistant* points x_i with $x_1 = \frac{1}{2}$ and $x_N = \frac{3}{2}$, write a function that can evaluate the interpolating polynomial p_{N-1} for these x_i and $y_i = \frac{1}{x_i}$. Plot $f(x)$, $p_{N-1}(x)$, and $|f(x) - p_{N-1}(x)|$ and visually evaluate the size of the error for a few different values of N. Plot the maximal error as a function of N for a number of values of N in the range $1 \dots 30$. Describe the qualitative behavior you observe as N becomes larger.

(b) Repeat the experiment but with the interpolation points chosen as the roots of the Chebyshev polynomial (but of course shifted to the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$).

(20 points)

Problem 4 (Runge's example for the Lagrange interpolation error). Repeat Problem 3(a) for the function $f(x) = \frac{1}{1+x^2}$ and the interval $[a, b] = [-5, 5]$. If you use equidistant points x_i on this interval, you should find that the error

$$\max_{-5 \le x \le 5} |f(x) - p_{N-1}(x)|$$

grows without bound as $N \to \infty$. This is what is called "Runge's example" and was given as early as 1901 to demonstrate that even for functions that are analytic on the real line, the interpolant may not converge. **(15 points)**

Problem 5 (Piecewise linear interpolation). Take again the function $f(x) = \frac{1}{1+x^2}$ and the interval $[a, b] = [-5, 5]$. Use again N equidistant points on this interval, but instead of finding one *global* polynomial $p_{N-1}(x)$, now use a *piecewise* interpolant $p_1^h(x)$ that is linear on each of the intervals $[x_i, x_{i+1}]$.[1] Here, the index h on p_1^h indicates that we are working with sub-intervals of length $h = x_{i+1} - x_i = \frac{1}{N-1}$, and the index 1 represents the polynomial degree we use for the interpolant on each sub-interval.

Plot $f(x)$, $p_1^h(x)$, and $|f(x) - p_1^h(x)|$ and visually evaluate the maximal size of the error for a few different values of N. Plot the maximal error as a function of N for a number of values of N in the range $1 \dots 1000$. Describe the qualitative behavior you observe as N becomes larger. **(15 points)**

Problem 6 (Piecewise quadratic interpolation). Repeat the previous problem with odd numbers N of equidistant interpolation points x_i where you now define the small intervals as $[x_1, x_3], [x_3, x_5], \dots$ and on each of these intervals the interpolant $p_2^h(x)$ is defined by finding the *quadratic* interpolant that interpolates x_i, x_{i+1}, x_{i+2}.

Plot $f(x)$, $p_2^h(x)$, and $|f(x) - p_2^h(x)|$ and visually evaluate the maximal size of the error for a few different values of N. Plot the maximal error as a function of N for a number of values of N in the range $1 \dots 1000$. Compare with the results of the previous problem. Can you explain the behavior of the error for the two approximants p_1^h, p_2^h theoretically? **(25 points)**

[1] Note that the overall costs – namely, N evaluations of the function $f(x)$ – is exactly the same as for the one global polynomial, if one assumes that evaluating $f(x)$ is far more expensive than evaluating $p_{N-1}(x)$. This may not be true for this particular choice of $f(x)$, but then this is only an example after all: the functions $f(x)$ for which we want to use this procedure can be thought of as "expensive" to evaluate.