Part 23

Optimal Control: Examples

Definition of optimal control problems

Commonly understood definition of optimal control problems:

Let

- X a space of time-dependent functions
- Q a space of control parameters, time dependent or not
- $f: X \times Q \to \mathbb{R}$ a continuous *functional* on X and Q
- $L: X \times Q \to Y$ continuous operator on X mapping into a space Y
- $g: X \to Z_x$ continuous operator on X mapping into a space Z_x
- $h: Q \to Z_q$ continuous operator on Q mapping into a space Z_q

Then the problem

$$\begin{array}{lllll} \min_{x=x(t)\in X,\, q\in Q} & f(x(t),q) & & & \\ \text{such that} & L(x(t),q) & =0 & & \forall t \in [t_i, t_f] \\ & g(x(t)) & \geq 0 & & \forall t \in [t_i, t_f] \\ & h(q) & \geq 0 & & \end{array}$$

is called an *optimal control problem*.

Definition of optimal control problems

Remark:

For existence and uniqueness of solutions of the problem

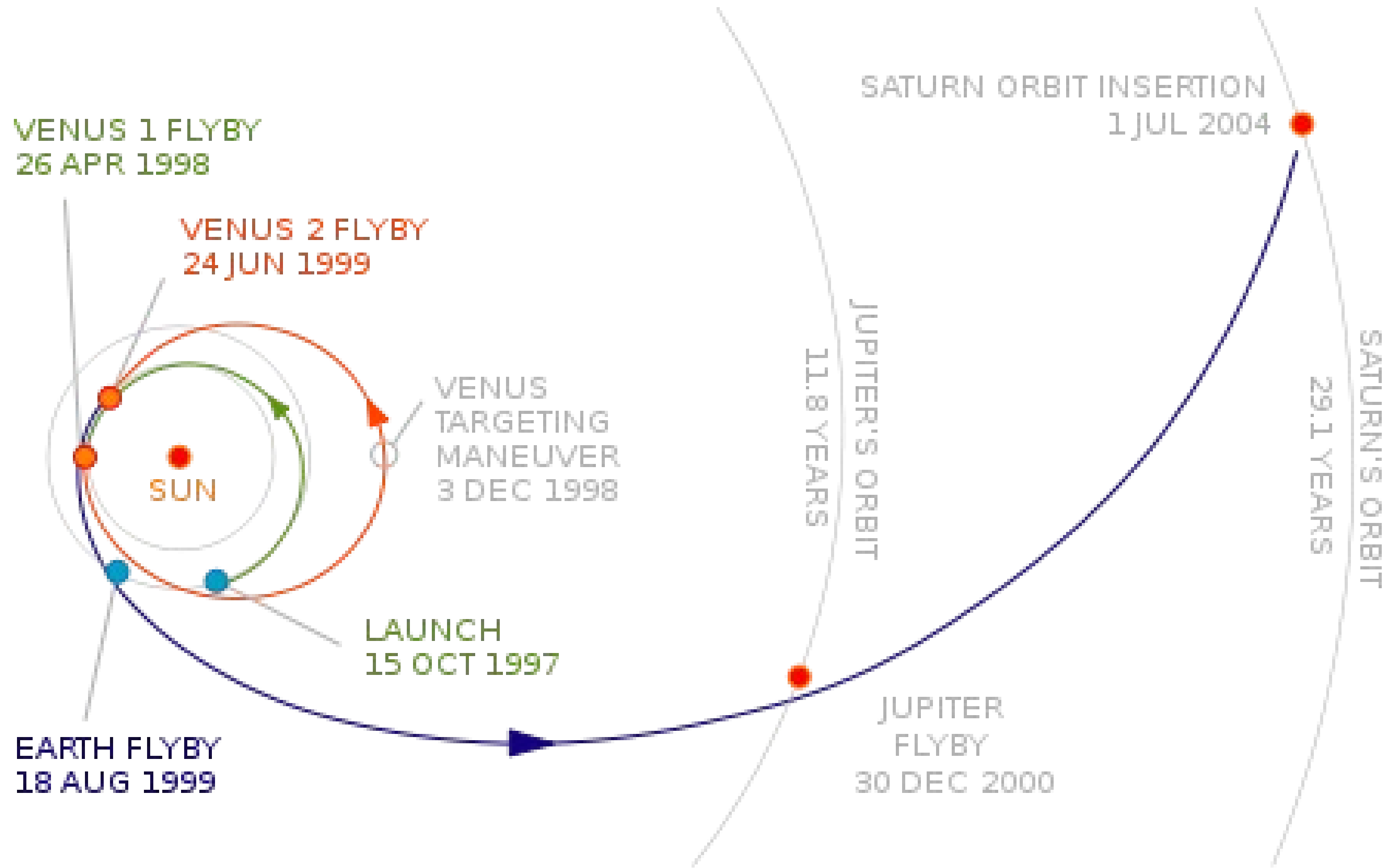
$$
\begin{array}{lll}
\min_{x=x(t)\in X,\, q\in Q} & f(x(t),q) & \\
\text{such that} & L(x(t),q)=0 & \forall t\in[t_i, t_f] \\
 & g(x(t)) \quad \geq 0 & \forall t\in[t_i, t_f] \\
 & h(q) \qquad \geq 0 &
\end{array}
$$

one will need convexity properties of *f,L,g,h*.

In order to state optimality conditions, we will in general also require certain differentiability properties.

Example 1: Trajectory planning

The trajectory of the Cassini space probe from Earth to Saturn:



Goal: We want to get from A to B using the least amount of fuel, in the least amount of time, ..., subject to Newton's law.

Example 1: Trajectory planning

Version 1: Minimal energy trajectory

- $X=\{x(t): x\in H^1([0,T])^3\}=\{x(t): x(t)\in L^2([0,T])^3, \dot{x}(t)\in L^2((0,T))^3\}$
- $Q=\{u(t): u\in L^\infty([0,T])^3\} \subset L^2([0,T])^3$
- $f: Q\to \mathbb{R}$
- $L: X\times Q\to Y, \qquad Y=H^{-1}([0,T])^3=\left(H^1([0,T])^3\right)^*$
- $g: X\to Z_x=\mathbb{R}^3\times\mathbb{R}^3$
- $h: Q\to Z_q=L^\infty([0,T])^3$

Then the problem is as follows:

$$
\begin{array}{lll}
\min_{x=x(t)\in X,\, q\in Q} & \int_0^T |u(t)| & \\
\text{such that} & m\ddot{x}(t)-ku(t)=0 & \forall\, t\in[0,T] \\
 & x(0)=\text{Earth}, \quad x(T)=\text{Saturn} & \\
 & u_{\max}-|u(t)| \quad \geq 0 & \forall\, t\in[0,T]
\end{array}
$$

Example 1: Trajectory planning

Remark 1:

A more realistic formulation would take into account that the mass of the space ship diminishes as fuel is burnt:

$$m = m(t) = m(0) - \int_0^t |u(t)|$$

Remark 2:

The formulation on the previous page is nonlinear because of the absolute values $|u(t)|$. The objective function can be made linear by using the following reparameterisation:

$$u(t) = \hat{u}(t)\theta(t), \qquad \hat{u}(t) \in \mathbb{R}_0^+, \quad \theta \in S^2$$

On the other hand, the ODE constraint will then be nonlinear (a complication that is usually easier to handle).

Example 1: Trajectory planning

Version 2: Minimal time trajectory

- $X = H^1([0,T])^3$
- $Q = [u(t), T] = L^\infty([0,T])^3 \times \mathbb{R}_0^+$
- $f : Q \to \mathbb{R}$
- $L : X \times Q \to Y$
- $g : X \to Z_x = \mathbb{R}^3 \times \mathbb{R}^3$
- $h : Q \to Z_q = L^\infty([0,T])^3$

Then the problem is as follows:

$$
\begin{array}{lllll}
\min_{x=x(t)\in X,\, q\in Q} & T & & & \\
\text{such that} & m\ddot{x}(t) - k u(t) = 0 & & & \forall\, t \in [0,T] \\
& x(0) = \text{Earth}, & & x(T) = \text{Saturn} & \\
& u_{\max} - |u(t)| & \geq 0 & \forall\, t \in [0,T] &
\end{array}
$$

Example 1: Trajectory planning

Version 3: Minimal thrust requirement trajectory

- $X = H^1([0,T])^3$
- $Q = [u(t), u_{\max}] = L^\infty([0,T])^3 \times \mathbb{R}_0^+$
- $f : Q \to \mathbb{R}$
- $L : X \times Q \to Y$
- $g : X \to Z_x = \mathbb{R}^3 \times \mathbb{R}^3$
- $h : Q \to Z_q = L^\infty([0,T])^3$

Then the problem is as follows:

$$
\begin{array}{lll}
\min_{x = x(t) \in X,\, q \in Q} & u_{\max} & \\
\text{such that} & m\ddot{x}(t) - k\,u(t) = 0 & \forall\, t \in [0,T] \\
& x(0) = \text{Earth}, \quad x(T) = \text{Saturn} & \\
& u_{\max} - |u(t)| \geq 0 & \forall\, t \in [0,T]
\end{array}
$$

Example 1: Trajectory planning

Remark 1: Similar problems appear in planning the paths of

- mobile robots
- air planes, manned or unmanned
- the arms of stationary robots (e.g. welding robots on assembly lines)
- braking a car without exceeding the maximal force the tires can transmit to the road

Remark 2: For some problems, $T=\infty$. These are called *infinite horizon* problems.

Example: Keeping a satellite or airship stationary at a given point above earth.

Example 2: Chemical reactors

State:
Concentrations $x_i(t)$ of chemical species $i=1...N$.

Controls:
Pressure $p(t)$, temperature $T(t)$.

Goals:

- Maximize output of a particular species
- Maximize purity
- Minimize cost
- Minimize time

Example 2: Chemical reactors

Version 1: Maximize yield of species *N*

$$\begin{aligned}
&\min_{x(t),\, p(t),\, T(t)} && -x_N(T) \\
&\text{such that} && \dot{x}(t) - f\left(x(t), p(t), T(t)\right) = 0 && \forall t \in [0, T] \\
& && x(0) = x_0 \\
& && p_0 \le p(t) \le p_1, \quad T_0 \le T(t) \le T_1 && \forall t \in [0, T]
\end{aligned}$$

Example 2: Chemical reactors

Version 2: Minimize reaction time, subject to minimum yield constraints:

$$
\begin{array}{lll}
\min_{x(t),\, p(t),\, T(t)} & T & \\
\text{such that} & \dot{x}(t) - f(x(t), p(t), T(t)) = 0 & \forall t \in [0, T] \\
 & x(0) = x_0 & \\
 & p_0 \le p(t) \le p_1, \quad T_0 \le T(t) \le T_1 & \forall t \in [0, T] \\
 & x_N \ge x_{N,\min} &
\end{array}
$$

Example 2: Chemical reactors

Version 3: Minimize cost due to heat losses (heat loss factor alpha) and due to the cost of changing temperature by cooling/ heating (cost factor beta), subject to minimum yield constraints:

$$
\begin{array}{lll}
\min_{x(t),\, p(t),\, T(t)} & \int_0^T \alpha T(t) + \beta \left| \dot{T}(t) \right| & \\
\text{such that} & \dot{x}(t) - f(x(t), p(t), T(t)) = 0 & \forall t \in [0, T] \\
& x(0) = x_0 & \\
& p_0 \le p(t) \le p_1, \quad T_0 \le T(t) \le T_1 & \forall t \in [0, T] \\
& x_N \ge x_{N,\min} &
\end{array}
$$

Part 24

Optimal control:
The shooting method

The solution operator

Definition:

*S*tate and control variables are connected by an ODE:

$$\dot{x}(t) - f(x(t), q) = 0 \qquad \forall t \in [t_i, t_f]$$
$$x(t_i) = g(x_0, q)$$

Let *x(t)* be the solution for a given set of control variables *q*. Then define

$$S(q, x_0, t_i, t) := x(t)$$

In other words: *S* is the operator that given controls and initial data provides the value of the corresponding solution of the ODE at time *t*. We call *S* the *solution operator*.

Note: If the ODE is complicated, then *S* is a purely theoretical construct, though it can be approximated numerically.

The solution operator

Corollary:

Consider the optimal control problem

$$\min_{x(t),q} \ \frac{1}{2}\left(x(t_f)-x_{\text{desired}}\right)^2$$
$$\dot{x}(t)-f(x(t),q)=0 \qquad \forall\, t\in[t_i,t_f]$$
$$x(t_i)=g(x_0,q)$$

It is equivalent to the problem

$$\min_{q} \ \frac{1}{2}\left(S(q,x_0,t_i,t_f)-x_{\text{desired}}\right)^2$$

Note 1: Similar reformulations are trivially available if the objective function has a different form or if there are constraints.

Note 2: If we can represent *S* and its derivatives, then we can apply Newton's method (or any other optimization method) to the reformulated problem.

The shooting method

Algorithm:

Start from the formulation:

$$\min_q \ \frac{1}{2}\left(S\left(q, x_0, t_i, t_f\right) - x_{\text{desired}}\right)^2$$

The shooting method is an iterative procedure with the following steps:

- Start with a certain control value *q*
- Compute the trajectory *S(q,...)* for this control value
- If we “overshoot” the goal, then do the same again with a smaller value of *q*
- If we “undershoot” the goal, try a larger value of *q*
- Iterate until we have the solution we were looking for

The shooting method: An example

Example: Charged particles in a magnetic field

Charged particles moving in a magnetic field follow the Lorentz force:

$$m\ddot{x}(t) = e\,\dot{x}(t) \times B(x(t), t)$$

Here:

– *e* charge of the particle

– *B(x(t),t)* magnetic field at *x(t)* and *t*

Assume the direction of *B(x,t)* is constant but that the magnitude is adjustable.

Goal: Given *x(0), d/dt x(0)*, find *B* for which *x(t)* passes through location x_{desired}.

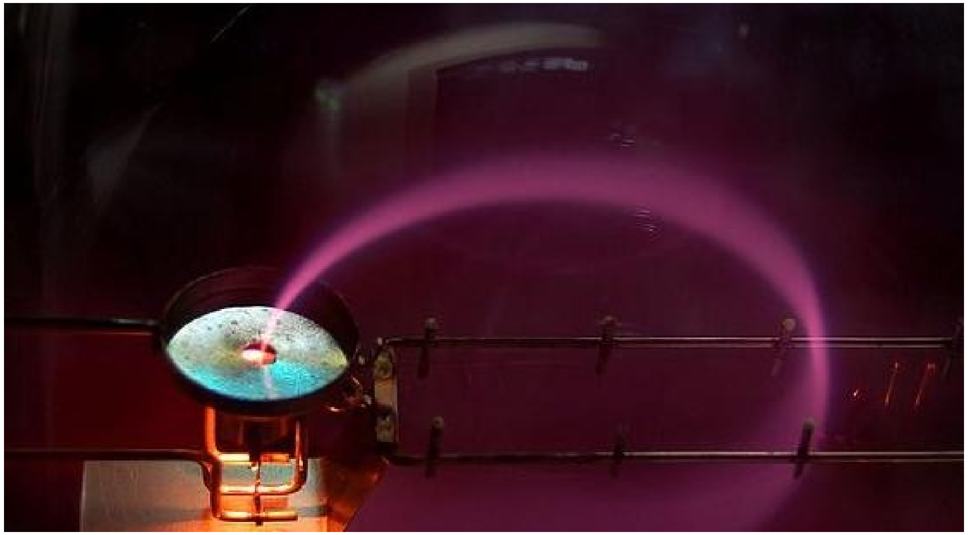
Formulation:

$$\min_{x(t),B,T} \ \frac{1}{2}\left(x(T) - x_{\text{desired}}\right)^2$$

$$m\ddot{x}(t) - e\,x(t) \times B = 0$$

$$x(0) = x_0$$

$$\dot{x}(0) = v_0$$



The shooting method: An example

Example: Charged particles in a magnetic field

For

$$m\ddot{x}(t) = e\,\dot{x}(t) \times B, \qquad x(0) = 0, \quad \dot{x}(0) = \begin{pmatrix} 0 \\ v_0 \end{pmatrix}$$

and if *B* is in *z*-direction, the exact trajectory is:

$$x(t) = r \begin{pmatrix} 1 - \cos \omega t \\ \sin \omega t \end{pmatrix}$$

where

$$r = \frac{m v_0}{e\|B\|}, \quad \omega = \frac{v_0}{r} = \frac{e\|B\|}{m}$$

Then the solution operator is:

$$S(B, 0, t) = r \begin{pmatrix} 1 - \cos \omega t \\ \sin \omega t \end{pmatrix}$$

The shooting method: An example

Example: Charged particles in a magnetic field

Now: Restate the original problem

$$\min_{x(t),B,T} \ \frac{1}{2}\left(x(T)-x_{\text{desired}}\right)^2$$
$$m\ddot{x}(t)-e\,x(t)\times B=0$$
$$x(0)=x_0$$
$$\dot{x}(0)=v_0$$

as:

$$\min_{B,T} \ \frac{1}{2}\left|S(B,0,T)-x_{\text{desired}}\right|^2 \ = \ \frac{1}{2}\left(r\begin{pmatrix}1-\cos\omega T\\ \sin\omega T\end{pmatrix}-x_{\text{desired}}\right)^2$$

Note: This is a nonlinear optimization problem in two variables (*B*,*T*) that we can solve with any of the usual methods.

The shooting method: Practical implementation

Consider the optimal control problem with control constraints:

$$
\begin{aligned}
\min_{x(t),q}\ & F(x(t),q) \\
& \dot{x}(t) - f(x(t),q) = 0 \qquad \forall\, t \in [t_i, t_f] \\
& x(t_i) = g(x_0, q) \\
& h(q) \geq 0
\end{aligned}
$$

It is equivalent to the problem

$$
\begin{aligned}
\min_{q}\ & F(S(q, x_0, t_i, t), q) \\
& h(q) \geq 0
\end{aligned}
$$

Using the techniques we know (e.g. the active set method, barrier methods, etc), we can solve this problem.

However: We need first and second *derivatives* of *F* with respect to *q*!

The shooting method: Computing derivatives

By the chain rule, we have

$$\frac{d}{dq_i} F(S(q, x_0, t_i, t), q)$$
$$= \nabla_s F(S(q, x_0, t_i, t), q) \frac{d}{dq_i} S(q, x_0, t_i, t) + \frac{\partial}{\partial q_i} F(S(q, x_0, t_i, t), q)$$

That is, to compute derivatives of *F*, we need derivatives of *S*. To compute these, remember that

$$S(q, x_0, t_i, t) = x(t)$$

where $x(t)=x_q(t)$ solves the ODE for the given *q*:

$$\dot{x}(t) - f(x(t), q) = 0 \qquad \forall t \in [t_i, t_f]$$
$$x(t_i) = g(x_0, q)$$

The shooting method: Computing derivatives

By definition:

$$\frac{d}{dq_i} S(q, x_0, t_i, t) = \lim_{\epsilon \to 0} \frac{S(q+\epsilon e_i, x_0, t_i, t) - S(q, x_0, t_i, t)}{\epsilon}$$

Consequently, we can approximate derivatives using the formula

$$\frac{d}{dq_i} S(q, x_0, t_i, t) \approx \frac{S(q+\delta e_i, x_0, t_i, t) - S(q, x_0, t_i, t)}{\delta} = \frac{x_{q+\delta e_i}(t) - x_q(t)}{\delta}$$

for a finite $\delta > 0$. Note that $x_q(t)$ and $x_{q+\delta e_i}(t)$ solve the ODEs

$$\dot{x}_q(t) - f(x_q(t), q) = 0 \qquad x_q(t_i) = g(x_0, q)$$

$$\dot{x}_{q+\delta e_i}(t) - f(x_{q+\delta e_i}(t), q+\delta e_i) = 0 \qquad x_{q+\delta e_i}(t_i) = g(x_0, q+\delta e_i)$$

The shooting method: Computing derivatives

Corollary:

To compute $\nabla_q F(S(q,x_0,t_i,t),q)$ we need to compute

$$\nabla_q S(q,x_0,t_i,t)$$

For $q \in \mathbb{R}^n$, this requires the solution of *n+1* ordinary differential equations:

- For the given *q*:

$$\dot{x}_q(t) - f(x_q(t),q) = 0$$
$$x_q(t_i) = g(x_0,q)$$

- Perturbed in directions *i=1...n*:

$$\dot{x}_{q+\delta e_i}(t) - f(x_{q+\delta e_i}(t), q+\delta e_i) = 0$$
$$x_{q+\delta e_i}(t_i) = g(x_0, q+\delta e_i)$$

The shooting method: Computing derivatives

Practical considerations 1:

When computing finite difference approximations

$$\frac{d}{d q_i} S(q, x_0, t_i, t) \approx \frac{S(q+\delta e_i, x_0, t_i, t) - S(q, x_0, t_i, t)}{\delta} = \frac{x_{q+\delta e_i}(t) - x_q(t)}{\delta}$$

how should we choose the step length *δ*?

δ must be small enough to yield a good approximation to the exact derivative but large enough so that floating point roundoff does not affect the accuracy!

Rule of thumb: If

- ϵ is the precision of floating point numbers
- $\hat{q}_i$ is a typical size of the *i*th control variable q_i

then choose $\delta = \sqrt{\epsilon}\,\hat{q}_i$.

The shooting method: Computing derivatives

Practical considerations 2:

The one-sided finite difference quotient

$$\frac{d}{dq_i} S(q, x_0, t_i, t) \approx \frac{S(q+\delta e_i, x_0, t_i, t) - S(q, x_0, t_i, t)}{\delta} = \frac{x_{q+\delta e_i}(t) - x_q(t)}{\delta}$$

is only first order accurate in δ, i.e.

$$\left| \frac{d}{dq_i} S(q, x_0, t_i, t) - \frac{S(q+\delta e_i, x_0, t_i, t) - S(q, x_0, t_i, t)}{\delta} \right| = O(\delta)$$

The shooting method: Computing derivatives

Practical considerations 2:

Improvement: Use two-sided finite difference quotients

$$\frac{d}{dq_i}S(q,x_0,t_i,t)\approx\frac{S(q+\delta e_i,x_0,t_i,t)-S(q-\delta e_i,x_0,t_i,t)}{2\delta}=\frac{x_{q+\delta e_i}(t)-x_{q-\delta e_i}(t)}{2\delta}$$

which is second order accurate in δ, i.e.

$$\left|\frac{d}{dq_i}S(q,x_0,t_i,t)-\frac{S(q+\delta e_i,x_0,t_i,t)-S(q-\delta e_i,x_0,t_i,t)}{2\delta}\right|=O(\delta^2)$$

Note:

The cost for this higher accuracy is *2n+1* ODE solves!

The shooting method: Computing derivatives

Practical considerations 3:

Approximating derivatives requires solving the ODEs

$$\dot{x}_q(t) - f(x_q(t), q) = 0$$
$$x_q(t_i) = g(x_0, q)$$

$$\dot{x}_{q+\delta e_i}(t) - f(x_{q+\delta e_i}(t), q+\delta e_i) = 0 \qquad i = 1 \dots n$$
$$x_{q+\delta e_i}(t_i) = g(x_0, q+\delta e_i)$$

If we can do that analytically, then good.

If we do this numerically, then numerical approximation introduces systematic errors related to

- the numerical method used
- the time mesh (i.e. the collection of time step sizes) chosen

The shooting method: Computing derivatives

Practical considerations 3:

We gain the highest accuracy in the numerical solution of equations like

$$\dot{x}_q(t) - f(x_q(t), q) = 0$$
$$x_q(t_i) = g(x_0, q)$$

by choosing sophisticated adaptive time step, extrapolating multistep ODE integrators (e.g. RK45).

On the other hand, to get the best accuracy in evaluating

$$\frac{d}{dq_i} S(q, x_0, t_i, t) \approx \frac{x_{q+\delta e_i}(t) - x_q(t)}{\delta}$$

experience shows that we should use *predictable* integrators for all variables $x_q(t), x_{q+\delta e_i}(t)$ and use

- the same numerical method
- the same time steps
- no extrapolation

Practical considerations 3:

Thus, to solve the ODEs

$$\dot{x}_q(t) - f(x_q(t), q) = 0$$
$$x_q(t_i) = g(x_0, q)$$

$$\dot{x}_{q+\delta e_i}(t) - f(x_{q+\delta e_i}(t), q + \delta e_i) = 0$$
$$x_{q+\delta e_i}(t_i) = g(x_0, q + \delta e_i)$$

it is useful to solve them all at once as

$$\frac{d}{dt}\begin{pmatrix} x_q(t) \\ x_{q+\delta_1 e_1}(t) \\ \vdots \\ x_{q+\delta_n e_n}(t) \end{pmatrix} - \begin{pmatrix} f(x_q(t), q) \\ f(x_{q+\delta_1 e_1}(t), q + \delta_1 e_1) \\ \vdots \\ f(x_{q+\delta_n e_n}(t), q + \delta_n e_n) \end{pmatrix} = 0$$

$$\begin{pmatrix} x_q(t_i) \\ x_{q+\delta_1 e_1}(t_i) \\ \vdots \\ x_{q+\delta_n e_n}(t_i) \end{pmatrix} = \begin{pmatrix} g(x_0, q) \\ g(x_0, q + \delta_1 e_1) \\ \vdots \\ g(x_0, q + \delta_n e_n) \end{pmatrix}$$

The shooting method: Computing derivatives

Practical considerations 4:

For BFGS, we only need 1st derivatives of *F(S(q),q).* For a full Newton method we also need

$$\frac{d^2}{dq_i^2} S(q, x_0, t_i, t), \frac{d^2}{dq_i\, dq_j} S(q, x_0, t_i, t)$$

Again use finite difference methods:

$$\frac{d^2}{dq_i^2} S(q, x_0, t_i, t) \approx \frac{\frac{x_{q+\delta e_i}(t) - x_q(t)}{\delta} - \frac{x_q(t) - x_{q-\delta e_i}(t)}{\delta}}{\delta} = \frac{x_{q+\delta e_i}(t) - 2x_q(t) + x_{q-\delta e_i}(t)}{\delta^2}$$

$$\frac{d^2}{dq_i\, dq_j} S(q, x_0, t_i, t) \approx \frac{\frac{x_{q+\delta e_i+\delta e_j}(t) - x_{q-\delta e_i+\delta e_j}(t)}{2\delta} - \frac{x_{q+\delta e_i-\delta e_j}(t) - x_{q-\delta e_i-\delta e_j}(t)}{2\delta}}{2\delta}$$

 Note: The cost for this operation is 3^n ODE solves.

The shooting method: Practical implementation

Algorithm:

To solve

$$\begin{aligned} &\min_{x(t),q} \; F(x(t),q) \\ &\dot{x}(t) - f(x(t),q) = 0 \qquad \forall\, t \in [t_i, t_f] \\ &x(t_i) = g(x_0, q) \\ &h(q) \geq 0 \end{aligned}$$

reformulate it as

$$\begin{aligned} &\min_q \; F(S(q, x_0, t_i, t), q) \\ &h(q) \geq 0 \end{aligned}$$

Solve it using a known technique where

- by the chain rule $\nabla_q F(S,q) = F_S(S,q) \nabla_q S(q, x_0, t_i, t) + F_q(S,q)$ and similarly for second derivatives
- the quantities $\nabla_q S(q, x_0, t_i, t), \nabla_q^2 S(q, x_0, t_i, t)$ are approximated by finite difference quotients by solving multiple ODEs for different values of the control variable *q*

The shooting method: Practical implementation

Implementation (Newton method without line search; no attempt to compute ODE and its derivatives in synch):

```
function f(double[N] q) → double;
function grad_f(double[N] q) → double[N];
function grad_grad_f(double[N] q) → double[N][N];

function newton(double[N] q) → double[N]
{
    do {
        double[N] dq = - invert(grad_grad_f(q)) * grad_f(q);
        q = q + dq;
    } while (norm(grad_f(x)) > 1e-12);        // for example
    return q;
}
```

The shooting method: Practical implementation

Implementation (objective function only depends on $x(t_f)$):

```
function S(double[N] q, double t) → double[M]
{
   double[M] x = x0;
   double time = ti;
   while (time<t) {
            // explicit Euler method with fixed dt
      x = x + dt * rhs(x,q);
      time = time + dt;
   }
   return x;
}

function f(double[N] q) → double
{
   return objective_function(S(q, tf),q);
}
```

The shooting method: Practical implementation

Implementation (one-sided finite difference quotient):

```
function grad_f(double[N] q) → double[N]
{
    double[N] df = 0;
    for (i=1...N) {
        delta = 1e-8 * typical_q[i];
        double[N] q_plus = q;
        q_plus[i] = q[i] + delta;
        df[i] = (f(q_plus) – f(q)) / delta;
    }
    return df;
}
```

Part 25

Optimal control:
The multiple shooting method

Motivation

In the shooting method, we need to evaluate and differentiate the function

$$S(q, x_0, t_i, t) = x_q(t)$$

where $x_q(t)$ solves the ODE

$$\dot{x}(t) - f(x(t), q) = 0 \qquad \forall t \in [t_i, t_f]$$
$$x(t_i) = g(x_0, q)$$

Observation:

If the time interval $[t_i, t_f]$ is “long”, then S is often a strongly nonlinear function of q.

Consequence:

It is difficult to approximate S and derivatives numerically since errors grow like e^{LT}, where L is a Lipschitz constant of S and $T = t_f - t_i$.

Idea

Observation:

If the time interval $[t_i,t_f]$ is “long”, then S is often a strongly nonlinear function of q.

But then S should be less nonlinear on smaller intervals!

Idea:

While $S(q,x_0,t_i,t_f)$ is a strongly nonlinear function of q, we could introduce

$$t_i = t_0 < t_1 < \ldots < t_k < \ldots < t_K = t_f$$

and the functions $S(q,x_k,t_k,t_{k+1})$ should be less nonlinear and therefore simpler to approximate or differentiate numerically!

Multiple shooting

Outline:

To solve

$$\min_{x(t),q} \; F(x(t),q)$$
$$\dot{x}(t) - f(x(t),q) = 0 \qquad \forall\, t \in [t_i, t_f]$$
$$x(t_i) = g(x_0, q)$$
$$h(q) \geq 0$$

replace this problem by the following:

$$\min_{x^1(t),x^2(t),\dots,x^K(t),q} \; F(x(t),q)$$

$$\text{where} \quad x(t) := x^k(t) \qquad \forall\, t \in [t_{k-1}, t_k]$$

$$\text{such that} \quad \dot{x}^1(t) - f(x^1(t),q) = 0 \qquad \forall\, t \in [t_{0,} t_1]$$

$$x^1(t_i) = g(x_0, q)$$

$$\dot{x}^k(t) - f(x^k(t),q) = 0 \qquad \forall\, t \in [t_{k-1}, t_k], k = 2 \dots K$$

$$x^k(t_{k-1}) = x^{k-1}(t_{k-1})$$

$$h(q) \geq 0$$

Multiple shooting

Outline:

In this formulation, every x^k depends explicitly on x^{k-1}. We can decouple this:

$$\begin{aligned}
&\min_{x^1(t),\, x^2(t),\dots,\, x^K(t),\, \hat{x}_0^1,\dots,\, \hat{x}_0^K,\, q} \; F(x(t), q) \\
&\quad \text{where} \quad x(t) := x^k(t) \qquad\qquad \forall t \in [t_{k-1}, t_k] \\
&\text{such that} \quad \dot{x}^k(t) - f(x^k(t), q) = 0 \qquad \forall t \in [t_{k-1}, t_k],\ k = 1 \dots K \\
&\qquad\qquad x^k(t_{k-1}) = \hat{x}_0^k \\
&\qquad\qquad \hat{x}_0^1 - g(x_0, q) = 0 \\
&\qquad\qquad \hat{x}_0^k - x^{k-1}(t_{k-1}) = 0 \qquad \forall k = 2 \dots K \\
&\qquad\qquad h(q) \geq 0
\end{aligned}$$

Note: The "defect constraints" $\hat{x}_0^k - x^{k-1}(t_{k-1}) = 0$ need not be satisfied in intermediate iterations of Newton's method. They will only be satisfied at the solution, forcing *x(t)* to be continuous.

Multiple shooting

Outline with the solution operator:

By introducing the solution operator as before, the problem can be written as

$$\min_{\hat{x}_0^1,\dots,\hat{x}_0^K,q} \quad F\left(S\left(q,x_0,t_i,t\right),q\right)$$

$$\text{where} \quad S\left(q,x_0,t_i,t\right):=S\left(q,\hat{x}_0^k,t_{k-1},t\right) \qquad \forall t\in\left[t_{k-1},t_k\right]$$

$$\text{such that} \quad \hat{x}_0^1-g\left(x_0,q\right)=0$$

$$\hat{x}_0^k-S\left(q,\hat{x}_0^{k-1},t_{k-1},t_k\right)=0 \qquad \forall k=2\dots K$$

$$h\left(q\right)\geq 0$$

Note: We now only ever have to differentiate

$$S\left(q,\hat{x}_0^k,t_{k-1},t\right)$$

which integrates the ODE on the much shorter time intervals $[t_{k-1},t_k]$ and consequently is much less nonlinear.

Part 26

Optimal control: Introduction to the Theory

Preliminaries

Definition: A vector space is a set *X* of objects so that the following holds:

$$\forall x, y \in X: \qquad x + y \in X$$
$$\forall x \in X, \alpha \in \mathbb{R}: \quad \alpha x \in X$$

In addition, associativity, distributivity and commutativity of addition has to hold. There also need to be identity and null elements of addition and scalar multiplication.

Examples:

$$X = \mathbb{R}^N$$
$$X = C^0(0,T) = \{x(t) : x(t) \text{ is continuous on } (0,T)\}$$
$$X = C^1(0,T) = \{x(t) \in C^0(0,T) : x(t) \text{ is continuously differentiable on } (0,T)\}$$
$$X = L^2(0,T) = \left\{x(t) : \int_0^T |x(t)|^2 \, dt < \infty\right\}$$

Preliminaries

Definition: A scalar product is a mapping

$$\langle \cdot , \cdot \rangle : X \times Y \to \mathbb{R}$$

of a pair of vectors from (real) vector spaces *X,Y* into the real numbers. It needs to be linear. If *X*=*Y* and x=y, then it also needs to be positive or zero.

Examples:

$X = Y = \mathbb{R}^N$ $\qquad \langle x, y \rangle = \sum_{i=1}^{N} x_i y_i$

$\langle x, y \rangle = \sum_{i=1}^{N} \alpha_i x_i y_i \quad \text{with weights } 0 < \alpha_i < \infty$

$X = Y = l_2$ $\qquad \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$

$X = Y = L^2(0,T)$ $\qquad \langle x, y \rangle = \int_0^T x(t)\, y(t)\, dt$

Preliminaries

Definition: Given a space *X* and a scalar product

$$\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$$

we call *Y=X'* the *dual space of X* if *Y* is the largest space for which the scalar product above "makes sense".

Examples:

$X = \mathbb{R}^N$	$\langle x, y \rangle = \sum_{i=1}^{N} x_i y_i$	$Y = \mathbb{R}^N$
$X = C^0(0,T)$	$\langle x, y \rangle = \int_0^T x(t)\, y(t)\, dt$	$Y = S(0,T)$
$X = L^2(0,T)$	$\langle x, y \rangle = \int_0^T x(t)\, y(t)\, dt$	$Y = L^2(0,T)$
$X = L^p(0,T), 1 < p < \infty$	$\langle x, y \rangle = \int_0^T x(t)\, y(t)\, dt$	$Y = L^q(0,T), \frac{1}{p} + \frac{1}{q} = 1$

Lagrange multipliers for finite dimensional problems

Consider the following finite dimensional problem:

$$
\begin{array}{ll}
\min_{x\in\mathbb{R}^n} & f(x) \\
\text{such that} & g_1(x)=0 \\
 & g_2(x)=0 \\
 & \vdots \\
 & g_K(x)=0
\end{array}
$$

Definition: Let the Lagrangian be $L(x,\lambda)=f(x)-\sum_{i=1}^{K}\lambda_i g_i(x).$

Theorem: Under certain conditions on *f,g* the solution of above problem satisfies

$$
\begin{aligned}
\frac{\partial L}{\partial x_i}(x^*,\lambda^*)&=0, \qquad i=1,\dots,N \\
\frac{\partial L}{\partial \lambda_i}(x^*,\lambda^*)&=0, \qquad i=1,\dots,K
\end{aligned}
$$

Lagrange multipliers for optimal control problems

Consider the following optimal control problem:

$$\begin{array}{lll} \min_{x(t)} & f(x(t),t) & \\ \text{such that} & g(x(t),t)=0 & \forall t\in[0,T] \end{array}$$

Questions:

- What would be the corresponding Lagrange multiplier for such a problem?
- What would be the corresponding Lagrangian function?
- What are optimality conditions in this case?

Lagrange multipliers for optimal control problems

Formal approach: Take the problem

$$\begin{array}{lll} \min_{x(t)} & f(x(t),t) & \\ \text{such that} & g(x(t),t)=0 & \forall t \in [0,T] \end{array}$$

There are infinitely many constraints, one constraint for each time instant.

Following this idea, we would then have to replace

$$L(x,\lambda)=f(x)-\sum_{i=1}^{K}\lambda_i g_i(x).$$

by

$$L(x(t),\lambda(t))=f(x(t),t)-\int_0^T \lambda(t) g(x(t),t)\ dt$$

where we have one Lagrange multiplier for every time *t*: $\lambda(t)$.

Lagrange multipliers for optimal control problems

The "correct" approach: If we have a set of equations like

$$\begin{gathered} g_1(x)=0 \\ g_2(x)=0 \\ \vdots \\ g_K(x)=0 \end{gathered}$$

then we can write this as

$$\vec{g}(x)=0$$

which we can interpret as saying

$$\langle \vec{g}(x), h \rangle = 0 \qquad \forall h \in \mathbb{R}^K$$

Lagrange multipliers for optimal control problems

The "correct" approach: Likewise, if we have

$$g(x(t),t)=0$$

then we can interpret this in different ways:

- At *every possible time t* we want that *g(x(t),t)* equals zero
- The measure of the set *{t: g(x(t),t)≠0}* is zero ("almost all *t*")
- The integral $\int_0^T |g(x(t),t)|^2\,dt$ is zero
- If $g:X\times[0,T]\to V$ then *g(x(t),t)* is zero in *V*, i.e.

$$\langle g(x(t),t),h\rangle=\int_0^T g(x(t),t)\,h(t)\,dt=0 \qquad \forall h\in V'$$

Notes:

- The first and fourth statement are the same if $V=C^0([0,T])$
- The second and fourth statement are the same if $V=L^1([0,T])$
- The third and fourth statement are the same if $V=L^2([0,T])$

Lagrange multipliers for optimal control problems

In either case: Given

$$\begin{array}{ll} \min_{x(t)\in X} & f(x(t),t) \\ \text{such that} & g(x(t),t)=0 \end{array}$$

the Lagrangian is now

$$\begin{aligned} L(x(t),\lambda(t)) &= f(x(t),t) - \langle \lambda, g(x(t),t) \rangle \\ &= f(x(t),t) - \int_0^T \lambda(t) g(x(t),t) \, dt \end{aligned}$$

and

$$L: X \times V' \to \mathbb{R}$$

Optimality conditions for finite dimensional problems

Corollary: In view of the definition

$$\left\langle \nabla_x f(x), \xi \right\rangle = \lim_{\epsilon \to 0} \frac{f(x+\epsilon\xi) - f(x)}{\epsilon}$$

we can say that the gradient of a function $f : \mathbb{R}^K \to \mathbb{R}$ is a functional

$$\nabla_x f \ : \ \mathbb{R}^K \to \left(\mathbb{R}^K\right)'$$

In other words: The gradient of a function is an element in the dual space of its argument.

Note: For finite dimensional spaces, we can identify space and dual space. Alternatively, we can consider $\mathbb{R}^K$ as the space of column vectors with *K* elements and $\left(\mathbb{R}^K\right)'$ as the space of row vectors with *K* elements.
In either case, the dual product is well defined.

Optimality conditions for finite dimensional problems

Corollary: From above considerations it follows that for

$$
\begin{array}{ll}
\min_{x\in\mathbb{R}^n} & f(x) \\
\text{such that} & g_1(x)=0 \\
 & g_2(x)=0 \\
 & \vdots \\
 & g_K(x)=0
\end{array}
$$

we define

$$L(x,\lambda)=f(x)-\sum_{i=1}^{K}\lambda_i g_i(x)$$

where

$$L:\ \mathbb{R}^N\times\mathbb{R}^K\to\mathbb{R}$$

and

$$\nabla_x L:\ \mathbb{R}^N\times\mathbb{R}^K\to(\mathbb{R}^N)'$$
$$\nabla_\lambda L:\ \mathbb{R}^N\times\mathbb{R}^K\to(\mathbb{R}^K)'$$

Optimality conditions for finite dimensional problems

Summary: For the problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{such that} \quad & g_1(x)=0 \\ & g_2(x)=0 \\ & \vdots \\ & g_K(x)=0 \end{aligned}$$

we define

$$L(x,\lambda)=f(x)-\sum_{i=1}^{K}\lambda_i g_i(x).$$

The optimality conditions are then

$$\begin{aligned} \nabla_x L(x^*,\lambda^*)&=0 \quad \text{in } \mathbb{R}^N \\ \nabla_\lambda L(x^*,\lambda^*)&=0 \quad \text{in } \mathbb{R}^K \end{aligned}$$

or equivalently:

$$\begin{aligned} \langle \nabla_x L(x^*,\lambda^*),\xi\rangle&=0 \qquad \forall\, \xi\in\mathbb{R}^N \\ \langle \nabla_\lambda L(x^*,\lambda^*),\eta\rangle&=0 \qquad \forall\, \eta\in\mathbb{R}^K \end{aligned}$$

Optimality conditions for finite dimensional problems

Theorem: Under certain conditions on *f,g* the solution satisfies

$$\frac{\partial L}{\partial x_i}(x^*,\lambda^*)=0, \qquad i=1,\dots,N$$
$$\frac{\partial L}{\partial \lambda_i}(x^*,\lambda^*)=0, \qquad i=1,\dots,K$$

Note 1: These conditions can also be written as

$$\langle \nabla_x L(x^*,\lambda^*),\xi\rangle=0, \qquad \forall \xi\in\mathbb{R}^N$$
$$\langle \nabla_\lambda L(x^*,\lambda^*),\eta\rangle=0, \qquad \forall \eta\in\mathbb{R}^K$$

Note 2: This, in turn, can be written as follows:

$$\langle \nabla_x L(x^*,\lambda^*),\xi\rangle=\lim_{\epsilon\to 0}\frac{L(x^*+\epsilon\xi,\lambda^*)-L(x^*,\lambda^*)}{\epsilon}=0, \qquad \forall \xi\in\mathbb{R}^N$$
$$\langle \nabla_\lambda L(x^*,\lambda^*),\eta\rangle=\lim_{\epsilon\to 0}\frac{L(x^*,\lambda^*+\epsilon\eta)-L(x^*,\lambda^*)}{\epsilon}=0, \qquad \forall \eta\in\mathbb{R}^K$$

Optimality conditions for optimal control problems

Recall: For an optimal control problem

$$\begin{array}{ll} \min_{x(t)\in X} & f(x(t),t) \\ \text{such that} & g(x(t),t)=0 \end{array}$$

with

$$g: X\times\mathbb{R}\to V$$

we have defined the Lagrangian as

$$L(x(t),\lambda(t))=f(x(t),t)-\langle\lambda,g(x(t),t)\rangle$$

$$L: X\times V'\to\mathbb{R}$$

Optimality conditions for optimal control problems

Theorem: Under certain conditions on *f,g* the solution satisfies

$$\langle \nabla_x L(x^*,\lambda^*),\xi\rangle = 0, \qquad \forall \xi \in X$$
$$\langle \nabla_\lambda L(x^*,\lambda^*),\eta\rangle = 0, \qquad \forall \eta \in V$$

or equivalently

$$\int_0^T \nabla_x L(x^*(t),\lambda^*(t))\,\xi(t)\;dt = 0, \qquad \forall \xi \in X$$
$$\int_0^T \nabla_\lambda L(x^*(t),\lambda^*(t))\,\eta(t)\;dt = 0, \qquad \forall \eta \in V$$

Note: The derivative of the Lagrangian is defined as usual:

$$\langle \nabla_x L(x^*(t),\lambda^*(t)),\xi(t)\rangle = \lim_{\epsilon\to 0} \frac{L(x^*(t)+\epsilon\xi(t),\lambda^*(t)) - L(x^*(t),\lambda^*(t))}{\epsilon}$$
$$\langle \nabla_\lambda L(x^*(t),\lambda^*(t)),\eta(t)\rangle = \lim_{\epsilon\to 0} \frac{L(x^*(t),\lambda^*(t)+\epsilon\eta(t)) - L(x^*(t),\lambda^*(t))}{\epsilon}$$

Optimality conditions: Example 1

Example: Consider the rather boring problem

$$\begin{aligned} \min_{x(t)\in X} \quad & f(x(t),t)=\int_0^T x(t)\,dt \\ \text{such that} \quad & g(x(t),t)=x(t)-\psi(t)=0 \end{aligned}$$

for a given function $\psi(t)$. The solution is obviously $x(t)=\psi(t)$. Then the Lagrangian is defined as

$$\begin{aligned} L(x(t),\lambda(t)) &= \int_0^T x(t)\,dt-\langle\lambda(t),x(t)-\psi(t)\rangle \\ &= \int_0^T x(t)-\lambda(t)[x(t)-\psi(t)]\,dt \end{aligned}$$

and we can compute optimality conditions in the next step.

Optimality conditions: Example 1

Given

$$L(x(t),\lambda(t))=\int_0^T x(t)-\lambda(t)[x(t)-\psi(t)]\,dt$$

we can compute derivatives of the Lagrangian:

$$\begin{aligned}
&\langle \nabla_x L(x(t),\lambda(t)),\xi(t)\rangle \\
&\quad=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{\begin{array}{r} \int_0^T (x(t)+\epsilon\,\xi(t))-\lambda(t)[(x(t)+\epsilon\,\xi(t))-\psi(t)]\,dt \\ -\int_0^T x(t)-\lambda(t)[x(t)-\psi(t)]\,dt \end{array}\right\} \\
&\quad=\lim_{\epsilon\to 0}\frac{\int_0^T \epsilon\,\xi(t)-\lambda(t)[\epsilon\,\xi(t)]\,dt}{\epsilon} \\
&\quad=\int_0^T \xi(t)-\lambda(t)\,\xi(t)\,dt \\
&\quad=\int_0^T [1-\lambda(t)]\xi(t)\,dt
\end{aligned}$$

$$\langle \nabla_\lambda L(x(t),\lambda(t)),\eta(t)\rangle=\int_0^T -[x(t)-\psi(t)]\eta(t)\,dt$$

Optimality conditions: Example 1

Example: Consider the rather boring problem

$$\min_{x(t)\in X} \quad f(x(t),t)=\int_0^T x(t)\,dt$$
$$\text{such that} \quad g(x(t),t)=x(t)-\psi(t)=0$$

The optimality conditions are now

$$\left\langle \nabla_x L(x(t),\lambda(t)),\xi \right\rangle = \int_0^T [1-\lambda(t)]\xi(t)\,dt = 0 \qquad \forall\xi(t)$$
$$\left\langle \nabla_\lambda L(x(t),\lambda(t)),\eta \right\rangle = \int_0^T -[x(t)-\psi(t)]\eta(t)\,dt = 0 \qquad \forall\eta(t)$$

These can only be satisfied for

$$1-\lambda(t)=0, \qquad x(t)-\psi(t)=0, \qquad \forall\, 0\le t\le T$$

Optimality conditions: Example 2

Example: Consider the slightly more interesting problem

$$\begin{aligned} &\min_{x(t)\in X} && f(x(t),t)=\int_0^T x(t)^2\,dt \\ &\text{such that} && g(x(t),t)=\dot{x}(t)-t=0 \end{aligned}$$

The constraint allows all functions of the form $x(t)=a+\frac{1}{2}t^2$ for all constants *a*. Then the Lagrangian is defined as

$$\begin{aligned} L(x(t),\lambda(t)) &= \int_0^T x(t)^2\,dt - \langle \lambda(t),\dot{x}(t)-t\rangle \\ &= \int_0^T x(t)^2 - \lambda(t)[\dot{x}(t)-t]\,dt \end{aligned}$$

Note: For $x(t)=a+\frac{1}{2}t^2$ the objective function has the value

$$\int_0^T x(t)^2\,dt = \int_0^T \left[a+\frac{1}{2}t^2\right]^2 = \frac{1}{20}T^5+\frac{1}{3}aT^3+a^2T$$

which takes on its minimal value for $a=-\frac{1}{6}T^2$

Optimality conditions: Example 2

Given

$$L(x(t),\lambda(t))=\int_0^T x(t)^2-\lambda(t)[\dot{x}(t)-t]\,dt$$

we can compute derivatives of the Lagrangian:

$$\begin{aligned}
&\left\langle \nabla_x L(x(t),\lambda(t)),\xi(t)\right\rangle \\
&\quad=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{\begin{aligned}&\int_0^T (x(t)+\epsilon\,\xi(t))^2-\lambda(t)[\dot{x}(t)+\epsilon\,\dot{\xi}(t)-t]\,dt\\ &\qquad\qquad-\int_0^T x(t)^2-\lambda(t)[\dot{x}(t)-t]\,dt\end{aligned}\right\}\\
&\quad=\lim_{\epsilon\to 0}\frac{\int_0^T 2\,\epsilon\,x(t)\xi(t)+\epsilon^2\,\xi(t)^2-\lambda(t)[\epsilon\,\dot{\xi}(t)]\,dt}{\epsilon}\\
&\quad=\int_0^T 2\,x(t)\xi(t)-\lambda(t)\dot{\xi}(t)\,dt\\
&\quad=\int_0^T [2\,x(t)+\dot{\lambda}(t)]\xi(t)\,dt-[\lambda(t)\xi(t)]_{t=0}^T
\end{aligned}$$

$$\left\langle \nabla_\lambda L(x(t),\lambda(t)),\eta(t)\right\rangle=\int_0^T -[\dot{x}(t)-t]\eta(t)\,dt$$

Optimality conditions: Example 2

The optimality conditions are now

$$\left\langle \nabla_x L(x(t),\lambda(t)),\xi \right\rangle = \int_0^T [2x(t)+\dot{\lambda}(t)]\xi(t)\,dt - \left[\lambda(t)\xi(t)\right]_{t=0}^{T} = 0 \qquad \forall \xi(t)$$

$$\left\langle \nabla_\lambda L(x(t),\lambda(t)),\eta \right\rangle = \int_0^T -[\dot{x}(t)-t]\eta(t)\,dt = 0 \qquad \forall \eta(t)$$

From the second equation we can conclude that

$$\dot{x}(t)-t=0 \quad \rightarrow \quad x(t)=a+\frac{1}{2}t^2$$

On the other hand, the first equation yields

$$2x(t)+\dot{\lambda}(t)=0,\ \ \lambda(0)=0,\ \ \lambda(T)=0$$

Given the form of *x(t)*, the first of these three conditions can be integrated:

$$\lambda(t)=-2\mathrm{at}-\frac{1}{3}t^3+b$$

Enforcing boundary conditions then yields $b=0, a=-\frac{1}{6}T^2$

Optimality conditions: Example 3 – initial conditions

Theorem: Let $x \in C^1, f \in C^0$. If *x(t)* satisfies the initial value problem

$$\dot{x}(t) = f(x(t), t)$$
$$x(0) = x_0$$

then it also satisfies the "variational" equality

$$\int_0^T \left[\dot{x}(t) - f(x(t), t)\right] \lambda(t)\, dt + \left[x(0) - x_0\right] \lambda(0) = 0 \qquad \forall \lambda(t) \in C^0([0, T])$$

and vice versa.

Optimality conditions: Example 3 – initial conditions

Example: Consider the (again slightly boring) problem

$$\begin{aligned} \min_{x(t)\in X} \quad & f(x(t),t) = \int_0^T x(t)\,dt \\ \text{such that} \quad & \dot{x}(t) - t = 0 \\ & x(0) = 1 \end{aligned}$$

The constraint allows for only a single feasible point, $x(t) = 1 + \frac{1}{2}t^2$

The Lagrangian is now defined as

$$\begin{aligned} L(x(t),\lambda(t)) &= \int_0^T x(t)\,dt - \langle \lambda(t), \dot{x}(t) - t \rangle - [x(0)-1]\lambda(0) \\ &= \int_0^T x(t) - \lambda(t)[\dot{x}(t) - t]\,dt - \lambda(0)[x(0)-1] \end{aligned}$$

Optimality conditions: Example 3 – initial conditions

Given $L(x(t),\lambda(t)) = \int_0^T x(t) - \lambda(t)[\dot{x}(t) - t]\, dt - \lambda(0)[x(0)-1]$

we can compute derivatives of the Lagrangian:

$$
\begin{aligned}
&\langle \nabla_x L(x(t),\lambda(t)), \xi(t)\rangle \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left\{ \begin{aligned} &\int_0^T (x(t)+\epsilon\,\xi(t)) - \lambda(t)[\dot{x}(t)+\epsilon\,\dot{\xi}(t)-t]\, dt - \lambda(0)[x(0)+\epsilon\,\xi(0)-1] \\ &\qquad - \int_0^T x(t)-\lambda(t)[\dot{x}(t)-t]\, dt + \lambda(0)[x(0)-1] \end{aligned} \right\} \\
&= \lim_{\epsilon\to 0} \frac{\int_0^T \epsilon\,\xi(t) - \lambda(t)[\epsilon\,\dot{\xi}(t)]\, dt - \epsilon\,\lambda(0)\xi(0)}{\epsilon} \\
&= \int_0^T \xi(t) - \lambda(t)\dot{\xi}(t)\, dt - \lambda(0)\xi(0) \\
&= \int_0^T [1+\dot{\lambda}(t)]\xi(t)\, dt - [\lambda(t)\,\xi(t)]_{t=0}^T - \lambda(0)\xi(0) \\
&= \int_0^T [1+\dot{\lambda}(t)]\xi(t)\, dt - \lambda(T)\,\xi(T)
\end{aligned}
$$

$$
\langle \nabla_\lambda L(x(t),\lambda(t)), \eta(t)\rangle = \int_0^T -[\dot{x}(t)-t]\eta(t)\, dt - \eta(0)[x(0)-1]
$$

Optimality conditions: Example 3 – initial conditions

The optimality conditions are now

$$\left\langle \nabla_x L(x(t),\lambda(t)),\xi \right\rangle = \int_0^T [1+\dot{\lambda}(t)]\xi(t)\,dt - \lambda(T)\xi(T) \qquad = 0 \qquad \forall \xi(t)$$

$$\left\langle \nabla_\lambda L(x(t),\lambda(t)),\eta \right\rangle = \int_0^T -[\dot{x}(t)-t]\eta(t)\,dt - [x(0)-1]\eta(0) \quad = 0 \qquad \forall \eta(t)$$

From the second equation we can conclude that

$$\dot{x}(t) - t = 0$$
$$x(0) = 1$$

In other words: Taking the derivative of the Lagrangian with respect to the Lagrange multiplier gives us back the (initial value problem) constraint, just like in the finite dimensional case.

Note: The only feasible point of this constraint is of course

$$x(t) = 1 + \frac{1}{2}t^2$$

Optimality conditions: Example 3 – initial conditions

The optimality conditions are now

$$\left\langle \nabla_x L(x(t),\lambda(t)),\xi \right\rangle = \int_0^T [1+\dot{\lambda}(t)]\xi(t)\,dt - \lambda(T)\xi(T) \qquad = 0 \quad \forall \xi(t)$$

$$\left\langle \nabla_\lambda L(x(t),\lambda(t)),\eta \right\rangle = \int_0^T -[\dot{x}(t)-t]\eta(t)\,dt - [x(0)-1]\eta(0) \quad = 0 \quad \forall \eta(t)$$

From the first equation we can conclude that

$$1+\dot{\lambda}(t)=0$$
$$\lambda(T)=0$$

in much the same way as we could obtain the initial value problem for *x(t)*.

Note: This is a *final value problem* for the Lagrange multiplier! Its solution is

$$\lambda(t)=T-t$$

Optimality conditions: Example 4 – initial conditions

Note: If the objective function had been nonlinear, then the equation for *λ(t)* would contain *x(t)* but still be linear in *λ(t)*.

Example: Consider the (again slightly boring) variant of the same problem

$$\begin{aligned} \min_{x(t)\in X} \quad & f(x(t),t)=\int_0^T \frac{1}{2}x(t)^2\,dt \\ \text{such that} \quad & \dot{x}(t)-t=0 \\ & x(0)=1 \end{aligned}$$

The constraint allows for only a single feasible point, $x(t)=1+\frac{1}{2}t^2$

The Lagrangian is now defined as

$$L(x(t),\lambda(t))=\int_0^T \frac{1}{2}x(t)^2-\lambda(t)[\dot{x}(t)-t]\,dt-\lambda(0)[x(0)-1]$$

Optimality conditions: Example 4 – initial conditions

Given

$$L(x(t),\lambda(t))=\int_0^T \frac{1}{2}x(t)^2-\lambda(t)[\dot{x}(t)-t]\,dt-\lambda(0)[x(0)-1]$$

the derivatives of the Lagrangian are now:

$$\langle \nabla_x L(x(t),\lambda(t)),\xi\rangle=\int_0^T [x(t)+\dot{\lambda}(t)]\xi(t)\,dt-\lambda(T)\xi(T)$$

$$\langle \nabla_\lambda L(x(t),\lambda(t)),\eta\rangle=\int_0^T -[\dot{x}(t)-t]\eta(t)\,dt-\eta(0)[x(0)-1]$$

Optimality conditions: Example 4 – initial conditions

The optimality conditions are now

$$\left\langle \nabla_x L(x(t), \lambda(t)), \xi \right\rangle = \int_0^T [x(t) + \dot{\lambda}(t)] \xi(t)\, dt - \lambda(T) \xi(T) \quad = 0 \quad \forall \xi(t)$$

$$\left\langle \nabla_\lambda L(x(t), \lambda(t)), \eta \right\rangle = \int_0^T -[\dot{x}(t) - t] \eta(t)\, dt - [x(0) - 1] \eta(0) \quad = 0 \quad \forall \eta(t)$$

From the second equation we can again conclude that

$$\dot{x}(t) - t = 0$$
$$x(0) = 1$$

with solution

$$x(t) = 1 + \frac{1}{2} t^2$$

Optimality conditions: Example 4 – initial conditions

The optimality conditions are now

$$\left\langle \nabla_x L(x(t),\lambda(t)),\xi \right\rangle = \int_0^T [x(t)+\dot{\lambda}(t)]\xi(t)\,dt - \lambda(T)\xi(T) \qquad = 0 \qquad \forall\,\xi(t)$$

$$\left\langle \nabla_\lambda L(x(t),\lambda(t)),\eta \right\rangle = \int_0^T -[\dot{x}(t)-t]\eta(t)\,dt - [x(0)-1]\eta(0) \quad = 0 \qquad \forall\,\eta(t)$$

From the first equation we can now conclude that

$$x(t)+\dot{\lambda}(t)=0$$
$$\lambda(T)=0$$

Note: This is a *linear final value problem* for the Lagrange multiplier.

Given the form of *x(t)*, we can integrate the first equation:

$$\lambda(t) = -t - \frac{1}{6}t^3 + a$$

Together with the final condition, we obtain

$$\lambda(t) = -t - \frac{1}{6}t^3 + T + \frac{1}{6}T^3$$

Optimality conditions: Preliminary summary

Summary so far: Consider the (not very interesting) case where the constraints completely determine the solution, i.e. without any control variables:

$$\begin{aligned} \min_{x(t)\in X} \quad & f(x(t),t) = \int_0^T F(x(t),t)\, dt \\ \text{such that} \quad & \dot{x}(t) - g(x(t),t) = 0 \\ & x(0) = x_0 \end{aligned}$$

Then the optimality conditions read in “variational form”:

$$\langle \nabla_x L(x(t),\lambda(t)), \xi \rangle = \int_0^T [F_x(x(t),t) + g_x(x(t),t) + \dot{\lambda}(t)]\xi(t)\, dt - \lambda(T)\xi(T) = 0$$

$$\langle \nabla_\lambda L(x(t),\lambda(t)), \eta \rangle = \int_0^T -[\dot{x}(t) - g(x(t),t)]\eta(t)\, dt - [x(0) - x_0]\eta(0) \ = 0$$

$$\forall \xi(t), \eta(t)$$

Optimality conditions: Preliminary summary

Summary so far: Consider the (not very interesting) case where the constraints completely determine the solution, i.e. without any control variables:

$$\begin{aligned} \min_{x(t)\in X} \quad & f(x(t),t)=\int_0^T F(x(t),t)\,dt \\ \text{such that} \quad & \dot{x}(t)-g(x(t),t)=0 \\ & x(0)=x_0 \end{aligned}$$

Then the optimality conditions read in “strong” form:

$$\begin{aligned} & \dot{x}(t)-g(x(t),t)=0 \qquad && \dot{\lambda}(t)=-F_x(x(t),t)-g_x(x(t),t) \\ & x(0)=x_0 && \lambda(T)=0 \end{aligned}$$

Note: Because *x(t)* does not depend on the Lagrange multiplier, the optimality conditions can be solved by first solving for *x(t)* as an initial value problem from 0 to *T* and in a second step solving the final value problem for *λ(t)* backward from *T* to 0.

Part 27

Optimal control: Theory

Optimality conditions for optimal control problems

Recap:

Let

- *X* a space of time-dependent functions
- *Q* a space of control parameters, time dependent or not
- $f : X \times Q \to \mathbb{R}$ a continuous *functional* on *X* and *Q*
- $L : X \times Q \to Y$ continuous operator on *X* mapping into a space *Y*
- $g : X \to Z_x$ continuous operator on *X* mapping into a space Z_x
- $h : Q \to Z_q$ continuous operator on *Q* mapping into a space Z_q

Then the problem

$$\begin{array}{lll} \min_{x=x(t)\in X,\, q\in Q} & f(x(t),q) & \\ \text{such that} & L(x(t),q)=0 & \forall t\in[t_i,t_f] \\ & g(x(t)) \geq 0 & \forall t\in[t_i,t_f] \\ & h(q) \geq 0 & \end{array}$$

 is called an *optimal control problem*.

Optimality conditions for optimal control problems

There are two important cases:

- The space of control parameters, *Q*, is a finite dimensional set

$$\begin{array}{lll} \min_{x=x(t)\in X,\, q\in Q=\mathbb{R}^n} & f(x(t),q) & \\ \text{such that} & L(x(t),q)=0 & \forall\, t\in[t_i,t_f] \\ & g(x(t)) \geq 0 & \forall\, t\in[t_i,t_f] \\ & h(q) \geq 0 & \end{array}$$

- The space of control parameters, *Q*, consists of time dependent functions

$$\begin{array}{lll} \min_{x=x(t)\in X,\, q\in Q} & f(x(t),q(t)) & \\ \text{such that} & L(x(t),q(t))=0 & \forall\, t\in[t_i,t_f] \\ & g(x(t),q(t)) \geq 0 & \forall\, t\in[t_i,t_f] \\ & h(q(t)) \geq 0 & \end{array}$$

The finite dimensional case

Consider the case of finite dimensional control variables *q*:

$$
\begin{aligned}
\min_{x(t)\in X,\, q\in\mathbb{R}^n} \quad & f(x(t),t,q)=\int_0^T F(x(t),t,q)\,dt \\
\text{such that} \quad & \dot{x}(t)-g(x(t),t,q)=0 \\
& x(0)=x_0(q)
\end{aligned}
$$

with

$$g: X\times\mathbb{R}\times\mathbb{R}^n\to V$$

Because the differential equation now depends on *q*, the feasible set is no longer just a single point. Rather, for every *q* there is a feasible *x(t)* if the ODE is solvable.

In this case, we have (all products are understood to be dot products):

$$L(x(t),q,\lambda(t))=\int_0^T F(x(t),t,q)\,dt-\langle\lambda,\dot{x}(t)-g(x(t),t,q)\rangle-\lambda(0)[x(0)-x_0(q)]$$

$$L: X\times\mathbb{R}^n\times V'\to\mathbb{R}$$

The finite dimensional case

Theorem: Under certain conditions on *f,g* the solution satisfies

$$\langle \nabla_x L(x^*, q^*, \lambda^*), \xi \rangle = 0, \qquad \forall \xi \in X$$

$$\langle \nabla_\lambda L(x^*, q^*, \lambda^*), \eta \rangle = 0, \qquad \forall \eta \in V$$

$$\langle \nabla_q L(x^*, q^*, \lambda^*), \rho \rangle = 0, \qquad \forall \rho \in (\mathbb{R}^n)' = \mathbb{R}^n$$

The first two conditions can equivalently be written as

$$\int_0^T \nabla_x L(x^*(t), q, \lambda^*(t))\, \xi(t)\; dt = 0, \qquad \forall \xi \in X$$

$$\int_0^T \nabla_\lambda L(x^*(t), q, \lambda^*(t))\, \eta(t)\; dt = 0, \qquad \forall \eta \in V$$

Note: Since *q* is finite dimensional, the following conditions are equivalent:

$$\langle \nabla_q L(x^*, q, \lambda^*), \rho \rangle = 0, \qquad \forall \rho \in (\mathbb{R}^n)' = \mathbb{R}^n$$

$$\nabla_q L(x^*, q, \lambda^*) = 0$$

The finite dimensional case

Corollary: Given the form of the Lagrangian,

$$L(x(t), q, \lambda(t)) = \int_0^T F(x(t), t, q) - \lambda(t)[\dot{x}(t) - g(x(t), t, q)]dt$$
$$-\lambda(0)[x(0) - x_0(q)]$$

the optimality conditions are equivalent to the following three sets of equations:

$$\dot{x}(t) = g(x(t), t, q), \qquad x(0) = x_0(q)$$

$$\dot{\lambda}(t) = -F_x(x(t), t, q) - g_x(x(t), t, q), \qquad \lambda(T) = 0$$

$$\int_0^T F_q(x(t), t, q) + \lambda(t) g_q(x(t), t, q) dt + \lambda(0) \frac{\partial x_0(q)}{\partial q} = 0$$

Remark: These are called the *primal*, *dual* and *control* equations, respectively.

The finite dimensional case

The optimality conditions for the finite dimensional case are

$$\dot{x}(t)=g(x(t),t,q), \qquad x(0)=x_0(q)$$

$$\dot{\lambda}(t)=-F_x(x(t),t,q)-g_x(x(t),t,q), \qquad \lambda(T)=0$$

$$\int_0^T F_q(x(t),t,q)+\lambda(t)\,g_q(x(t),t,q)\,dt+\lambda(0)\frac{\partial x_0(q)}{\partial q}=0$$

Note: The primal and dual equations are differential equations, whereas the control equation is a (in general nonlinear) algebraic equation. This should be enough to identify the two time-dependent functions and the finite dimensional parameter.

However: Since the control equation determines *q* for given primal and dual variables, we can no longer integrate the first equation forward and the second backward to solve the problem. *Everything is coupled now!*

The finite dimensional case: An example

Example: Throw a ball from height *h* with horizontal velocity v_x so that it lands as close as possible from *x=(1,0)* after one time unit:

$$\min_{\{x(t),v(t)\}\in X, q=\{h,v_x\}\in\mathbb{R}^2} \quad \frac{1}{2}\left(x(t)-\begin{pmatrix}1\\0\end{pmatrix}\right)^2 = \frac{1}{2}\int_0^T \left(x(t)-\begin{pmatrix}1\\0\end{pmatrix}\right)^2 \delta(t-1)\,dt$$

$$\text{such that} \qquad \dot{x}(t)=v(t) \qquad x(0)=\begin{pmatrix}0\\h\end{pmatrix}$$

$$\dot{v}(t)=\begin{pmatrix}0\\-1\end{pmatrix} \qquad v(0)=\begin{pmatrix}v_x\\0\end{pmatrix}$$

Then: $L(\{x(t),v(t)\},q,\{\lambda_x(t),\lambda_v(t)\})$

$$=\frac{1}{2}\int_0^T \left(x(t)-\begin{pmatrix}1\\0\end{pmatrix}\right)^2 \delta(t-1)\,dt - \langle \lambda_x, \dot{x}(t)-v(t)\rangle - \left\langle \lambda_v, \dot{v}(t)-\begin{pmatrix}0\\-1\end{pmatrix}\right\rangle$$

$$-\lambda_x(0)\left[x(0)-\begin{pmatrix}0\\h\end{pmatrix}\right]-\lambda_v(0)\left[v(0)-\begin{pmatrix}v_x\\0\end{pmatrix}\right]$$

The finite dimensional case: An example

From the Lagrangian

$$L\left(\{x(t),v(t)\},q,\{\lambda_x(t),\lambda_v(t)\}\right)$$
$$=\frac{1}{2}\int_0^T\left(x(t)-\begin{pmatrix}1\\0\end{pmatrix}\right)^2\delta(t-1)\,dt-\left\langle\lambda_x,\dot{x}(t)-v(t)\right\rangle-\left\langle\lambda_v,\dot{v}(t)-\begin{pmatrix}0\\-1\end{pmatrix}\right\rangle$$
$$-\lambda_x(0)\left[x(0)-\begin{pmatrix}0\\h\end{pmatrix}\right]-\lambda_v(0)\left[v(0)-\begin{pmatrix}v_x\\0\end{pmatrix}\right]$$

we get the optimality conditions:

- Derivative with respect to *x(t)*:

$$\int_0^T\left(x(t)-\begin{pmatrix}1\\0\end{pmatrix}\right)\xi_x(t)\delta(t-1)\,dt-\int_0^T\lambda_x(t)\dot{\xi}_x(t)\,dt-\lambda_x(0)\xi_x(0)=0 \qquad \forall\,\xi_x(t)$$

After integration by parts, we see that this is equivalent to

$$\left(x(t)-\begin{pmatrix}1\\0\end{pmatrix}\right)\delta(t-1)+\dot{\lambda}_x(t)=0 \qquad \lambda_x(T)=0$$

The finite dimensional case: An example

From the Lagrangian

$$L\left(\{x(t),v(t)\},q,\{\lambda_x(t),\lambda_v(t)\}\right)$$
$$=\frac{1}{2}\int_0^T\left(x(t)-\begin{pmatrix}1\\0\end{pmatrix}\right)^2\delta(t-1)\,dt-\left\langle\lambda_x,\dot{x}(t)-v(t)\right\rangle-\left\langle\lambda_v,\dot{v}(t)-\begin{pmatrix}0\\-1\end{pmatrix}\right\rangle$$
$$-\lambda_x(0)\left[x(0)-\begin{pmatrix}0\\h\end{pmatrix}\right]-\lambda_v(0)\left[v(0)-\begin{pmatrix}v_x\\0\end{pmatrix}\right]$$

we get the optimality conditions:

- Derivative with respect to *v(t)*:

$$\int_0^T\lambda_x(t)\xi_v(t)\,dt-\int_0^T\lambda_v(t)\dot{\xi}_v(t)\,dt-\lambda_v(0)\xi_v(0)=0\qquad\forall\xi_v(t)$$

After integration by parts, we see that this is equivalent to

$$\lambda_x(t)+\dot{\lambda}_v(t)=0\qquad\lambda_v(T)=0$$

The finite dimensional case: An example

From the Lagrangian

$$
\begin{aligned}
&L\left(\{x(t), v(t)\}, q, \{\lambda_x(t), \lambda_v(t)\}\right) \\
&= \frac{1}{2}\int_0^T \left(x(t) - \begin{pmatrix}1\\0\end{pmatrix}\right)^2 \delta(t-1)\,dt - \left\langle \lambda_x, \dot{x}(t) - v(t)\right\rangle - \left\langle \lambda_v, \dot{v}(t) - \begin{pmatrix}0\\-1\end{pmatrix}\right\rangle \\
&\quad - \lambda_x(0)\left[x(0) - \begin{pmatrix}0\\h\end{pmatrix}\right] - \lambda_v(0)\left[v(0) - \begin{pmatrix}v_x\\0\end{pmatrix}\right]
\end{aligned}
$$

we get the optimality conditions:

- Derivative with respect to $\lambda_x(t)$:

$$
\int_0^T \eta_x(t)\left[\dot{x}(t) - v(t)\right]dt - \eta_x(0)\left[x(0) - \begin{pmatrix}0\\h\end{pmatrix}\right] = 0 \qquad \forall\, \eta_x(t)
$$

This is equivalent to

$$
\dot{x}(t) - v(t) = 0 \qquad\qquad x(0) - \begin{pmatrix}0\\h\end{pmatrix} = 0
$$

The finite dimensional case: An example

From the Lagrangian

$$L\left(\{x(t), v(t)\}, q, \{\lambda_x(t), \lambda_v(t)\}\right)$$

$$= \frac{1}{2}\int_0^T \left(x(t) - \begin{pmatrix}1\\0\end{pmatrix}\right)^2 \delta(t-1)\,dt - \left\langle \lambda_x, \dot{x}(t) - v(t)\right\rangle - \left\langle \lambda_v, \dot{v}(t) - \begin{pmatrix}0\\-1\end{pmatrix}\right\rangle$$

$$-\lambda_x(0)\left[x(0) - \begin{pmatrix}0\\h\end{pmatrix}\right] - \lambda_v(0)\left[v(0) - \begin{pmatrix}v_x\\0\end{pmatrix}\right]$$

we get the optimality conditions:

- Derivative with respect to $\lambda_v(t)$:

$$\int_0^T \eta_v(t)\left[\dot{v}(t) - \begin{pmatrix}0\\-1\end{pmatrix}\right]dt - \eta_v(0)\left[v(0) - \begin{pmatrix}v_x\\0\end{pmatrix}\right] = 0 \qquad \forall\, \eta_v(t)$$

  This is equivalent to

$$\dot{v}(t) - \begin{pmatrix}0\\-1\end{pmatrix} = 0 \qquad v(0) - \begin{pmatrix}v_x\\0\end{pmatrix} = 0$$

The finite dimensional case: An example

From the Lagrangian

$$L(\{x(t),v(t)\},q,\{\lambda_x(t),\lambda_v(t)\})$$
$$=\frac{1}{2}\int_0^T\left(x(t)-\begin{pmatrix}1\\0\end{pmatrix}\right)^2\delta(t-1)\,dt-\left\langle\lambda_x,\dot{x}(t)-v(t)\right\rangle-\left\langle\lambda_v,\dot{v}(t)-\begin{pmatrix}0\\-1\end{pmatrix}\right\rangle$$
$$-\lambda_x(0)\left[x(0)-\begin{pmatrix}0\\h\end{pmatrix}\right]-\lambda_v(0)\left[v(0)-\begin{pmatrix}v_x\\0\end{pmatrix}\right]$$

we get the optimality conditions:

- Derivative with respect to the first control parameter *h*:

$$\lambda_{x,2}(0)=0$$

- Derivative with respect to the second control parameter v_x:

$$\lambda_{v,1}(0)=0$$

The finite dimensional case: An example

The complete set of optimality conditions is now as follows:

State equations:
(initial value problem)

$$\dot{x}(t) - v(t) = 0 \qquad x(0) - \begin{pmatrix} 0 \\ h \end{pmatrix} = 0$$

$$\dot{v}(t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0 \qquad v(0) - \begin{pmatrix} v_x \\ 0 \end{pmatrix} = 0$$

Adjoint equations:
(final value problem)

$$\left(x(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \delta(t-1) + \dot{\lambda}_x(t) = 0 \qquad \lambda_x(T) = 0$$

$$\lambda_x(t) + \dot{\lambda}_v(t) = 0 \qquad \lambda_v(T) = 0$$

Control equations:
(algebraic)

$$\lambda_{x,2}(0) = 0$$

$$\lambda_{v,1}(0) = 0$$

The finite dimensional case: An example

In this simple example, we can integrate the optimality conditions in time:

State equations:

(initial value problem)

$$\dot{x}(t) - v(t) = 0 \qquad x(0) - \begin{pmatrix} 0 \\ h \end{pmatrix} = 0$$

$$\dot{v}(t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0 \qquad v(0) - \begin{pmatrix} v_x \\ 0 \end{pmatrix} = 0$$

Solution:

$$v(t) = \begin{pmatrix} v_x \\ -t \end{pmatrix}$$

$$x(t) = \begin{pmatrix} v_x t \\ h - \frac{1}{2} t^2 \end{pmatrix}$$

The finite dimensional case: An example

In this simple example, we can integrate the optimality conditions in time:

Adjoint equations:
(final value problem)

$$\left(x(t)-\begin{pmatrix}1\\0\end{pmatrix}\right)\delta(t-1)+\dot{\lambda}_x(t)=0 \qquad \lambda_x(T)=0$$

$$\lambda_x(t)+\dot{\lambda}_v(t)=0 \qquad \lambda_v(T)=0$$

Solution:

$$\lambda_x(t)=-\left[x(1)-\begin{pmatrix}1\\0\end{pmatrix}\right] \quad \text{for } t<1 \qquad \lambda_v(t)=\left[x(1)-\begin{pmatrix}1\\0\end{pmatrix}\right]t \quad \text{for } t<1$$

$$\lambda_x(t)=0 \quad \text{for } t>1 \qquad \lambda_v(t)=0 \quad \text{for } t>1$$

Using what we found for x(1) previously:

$$\lambda_x(t)=-\begin{pmatrix}v_x-1\\ h-\frac{1}{2}\end{pmatrix} \quad \text{for } t<1 \qquad \lambda_v(t)=\begin{pmatrix}v_x-1\\ h-\frac{1}{2}\end{pmatrix}(t-1) \quad \text{for } t<1$$

$$\lambda_x(t)=0 \quad \text{for } t>1 \qquad \lambda_v(t)=0 \quad \text{for } t>1$$

The finite dimensional case: An example

In the final step, we use the control equations:

$$\lambda_{x,2}(0)=0$$

$$\lambda_{v,1}(0)=0$$

But we know that

$$\lambda_x(t)=-\begin{pmatrix} v_x-1 \\ h-\frac{1}{2} \end{pmatrix} \quad \text{for } t<1 \qquad\qquad \lambda_v(t)=\begin{pmatrix} v_x-1 \\ h-\frac{1}{2} \end{pmatrix}(t-1) \quad \text{for } t<1$$

Consequently, the solution is given by

$$h=\frac{1}{2}$$

$$v_x=1$$

The infinite dimensional case

Consider the case of a control variable *q(t)* that is a function (here, for example, a function in L^2):

$$\min_{x(t)\in X,\, q(t)\in L^2([0,T])} \quad f(x(t),t,q(t)) = \int_0^T F(x(t),t,q(t))\,dt$$

$$\text{such that} \quad \dot{x}(t) - g(x(t),t,q(t)) = 0$$

$$x(0) = x_0(q(0))$$

with

$$g : X \times \mathbb{R} \times \mathbb{R}^n \to V$$

In this case, we have

$$L(x(t),q(t),\lambda(t)) = \int_0^T F(x(t),t,q(t))\,dt - \left\langle \lambda, \dot{x}(t) - g(x(t),t,q(t)) \right\rangle$$
$$-\lambda(0)\left[x(0) - x_0(q(0))\right]$$

$$L : X \times L^2([0,T]) \times V' \to \mathbb{R}$$

The infinite dimensional case

Theorem: Under certain conditions on *f,g* the solution satisfies

$$\left\langle \nabla_x L(x^*, q^*, \lambda^*), \xi \right\rangle = 0, \qquad \forall \xi \in X$$
$$\left\langle \nabla_\lambda L(x^*, q^*, \lambda^*), \eta \right\rangle = 0, \qquad \forall \eta \in V$$
$$\left\langle \nabla_q L(x^*, q^*, \lambda^*), \rho \right\rangle = 0, \qquad \forall \rho \in L^2([0,T])' = L^2([0,T])$$

The first two conditions can equivalently be written as

$$\int_0^T \nabla_x L(x^*(t), q, \lambda^*(t))\, \xi(t)\ dt = 0, \qquad \forall \xi \in X$$
$$\int_0^T \nabla_\lambda L(x^*(t), q, \lambda^*(t))\, \eta(t)\ dt = 0, \qquad \forall \eta \in V$$

Note: Since *q* is is now a function, the third optimality condition is:

$$\int_0^T \nabla_q L(x^*(t), q, \lambda^*(t))\, \rho(t)\ dt = 0, \qquad \forall \rho \in L^2([0,T])$$

The infinite dimensional case

Corollary: Given the form of the Lagrangian,

$$L(x(t),q(t),\lambda(t))=\int_0^T F(x(t),t,q(t))\,dt-\langle\lambda,\dot{x}(t)-g(x(t),t,q(t))\rangle$$
$$-\lambda(0)[x(0)-x_0(q(0))]$$

the optimality conditions are equivalent to the following three sets of equations:

$$\dot{x}(t)=g(x(t),t,q(t)),\qquad x(0)=x_0(q(0))$$

$$\dot{\lambda}(t)=-F_x(x(t),t,q(t))-g_x(x(t),t,q(t)),\qquad \lambda(T)=0$$

$$F_q(x(t),t,q)+\lambda(t)g_q(x(t),t,q)=0,\qquad \lambda(0)\frac{\partial x_0(q)}{\partial q}=0$$

Remark: These are again called the *primal*, *dual* and *control* equations, respectively.

The infinite dimensional case

The optimality conditions for the infinite dimensional case are

$$\dot{x}(t) = g(x(t), t, q(t)), \qquad x(0) = x_0(q(0))$$

$$\dot{\lambda}(t) = -F_x(x(t), t, q(t)) - g_x(x(t), t, q(t)), \qquad \lambda(T) = 0$$

$$F_q(x(t), t, q) + \lambda(t) g_q(x(t), t, q) = 0, \qquad \lambda(0)\frac{\partial x_0(q)}{\partial q} = 0$$

Note 1: The primal and dual equations are differential equations, whereas the control equation is a (in general nonlinear) algebraic equation that has to hold for all times between 0 and *T*. This should be enough to identify the three time-dependent functions.

Note 2: Like for the finite dimensional case, all three equations are coupled and can not be solved one after the other.

The infinite dimensional case: An example

Example: Throw a ball from height 1. Use vertical thrusters so that the altitude follows the path *1+t²*:

$$\min_{\{x(t),\,v(t)\}\in X,\,q(t)\in L^2([0,T])} \quad \frac{1}{2}\int_0^T \left(x(t)-(1+t^2)\right)^2 dt$$

$$\text{such that} \qquad \dot{x}(t)=v(t) \qquad x(0)=1$$

$$\dot{v}(t)=-1+q(t) \qquad v(0)=0$$

Then:

$$L\left(\{x(t),v(t)\},q(t),\{\lambda_x(t),\lambda_v(t)\}\right)$$

$$=\frac{1}{2}\int_0^T \left(x(t)-(1+t^2)\right)^2 dt-\left\langle \lambda_x,\dot{x}(t)-v(t)\right\rangle-\left\langle \lambda_v,\dot{v}(t)-\left[-1+q(t)\right]\right\rangle$$

$$-\lambda_x(0)\left[x(0)-1\right]-\lambda_v(0)\left[v(0)-0\right]$$

The infinite dimensional case: An example

From the Lagrangian

$$L\left(\{x(t), v(t)\}, q(t), \{\lambda_x(t), \lambda_v(t)\}\right)$$
$$= \frac{1}{2}\int_0^T \left(x(t) - (1+t^2)\right)^2 dt - \left\langle \lambda_x, \dot{x}(t) - v(t) \right\rangle - \left\langle \lambda_v, \dot{v}(t) - [-1+q(t)] \right\rangle$$
$$-\lambda_x(0)\left[x(0) - 1\right] - \lambda_v(0)\left[v(0) - 0\right]$$

we get the optimality conditions:

- Derivative with respect to *x(t)*:

$$\int_0^T \left(x(t) - (1+t^2)\right)\xi_x(t)\,dt - \int_0^T \lambda_x(t)\dot{\xi}_x(t)\,dt - \lambda_x(0)\xi_x(0) = 0 \qquad \forall \xi_x(t)$$

  After integration by parts, we see that this is equivalent to

$$\left(x(t) - (1+t^2)\right) + \dot{\lambda}_x(t) = 0 \qquad \lambda_x(T) = 0$$

The infinite dimensional case: An example

From the Lagrangian

$$L\left(\{x(t), v(t)\}, q(t), \{\lambda_x(t), \lambda_v(t)\}\right)$$
$$= \frac{1}{2}\int_0^T \left(x(t) - (1+t^2)\right)^2 dt - \left\langle \lambda_x, \dot{x}(t) - v(t) \right\rangle - \left\langle \lambda_v, \dot{v}(t) - [-1 + q(t)] \right\rangle$$
$$- \lambda_x(0)\left[x(0) - 1\right] - \lambda_v(0)\left[v(0) - 0\right]$$

we get the optimality conditions:

- Derivative with respect to *v(t)*:

$$\int_0^T \lambda_x(t)\xi_v(t)\,dt - \int_0^T \lambda_v(t)\dot{\xi}_v(t)\,dt - \lambda_v(0)\xi_v(0) = 0 \qquad \forall \xi_v(t)$$

After integration by parts, we see that this is equivalent to

$$\lambda_x(t) + \dot{\lambda}_v(t) = 0 \qquad \lambda_v(T) = 0$$

The infinite dimensional case: An example

From the Lagrangian

$$
\begin{aligned}
&L\left(\left\{x(t), v(t)\right\}, q(t), \left\{\lambda_x(t), \lambda_v(t)\right\}\right) \\
&\quad = \frac{1}{2}\int_0^T \left(x(t) - \left(1+t^2\right)\right)^2 dt - \left\langle \lambda_x, \dot{x}(t) - v(t)\right\rangle - \left\langle \lambda_v, \dot{v}(t) - \left[-1+q(t)\right]\right\rangle \\
&\qquad - \lambda_x(0)\left[x(0)-1\right] - \lambda_v(0)\left[v(0)-0\right]
\end{aligned}
$$

we get the optimality conditions:

- Derivative with respect to $\lambda_x(t)$:

$$
\int_0^T \eta_x(t)\left[\dot{x}(t) - v(t)\right]dt - \eta_x(0)\left[x(0)-1\right] = 0 \qquad \forall\, \eta_x(t)
$$

This is equivalent to

$$
\dot{x}(t) - v(t) = 0 \qquad x(0) - 1 = 0
$$

The infinite dimensional case: An example

From the Lagrangian

$$L\left(\left\{x(t), v(t)\right\}, q(t), \left\{\lambda_x(t), \lambda_v(t)\right\}\right)$$
$$= \frac{1}{2}\int_0^T \left(x(t) - (1+t^2)\right)^2 dt - \left\langle \lambda_x, \dot{x}(t) - v(t) \right\rangle - \left\langle \lambda_v, \dot{v}(t) - [-1+q(t)] \right\rangle$$
$$- \lambda_x(0)\left[x(0) - 1\right] - \lambda_v(0)\left[v(0) - 0\right]$$

we get the optimality conditions:

- Derivative with respect to $\lambda_v(t)$:

$$\int_0^T \eta_v(t)\left[\dot{v}(t) - (-1+q(t))\right] dt - \eta_v(0)\left[v(0) - 0\right] = 0 \qquad \forall\, \eta_v(t)$$

This is equivalent to

$$\dot{v}(t) - (-1+q(t)) = 0 \qquad v(0) = 0$$

The infinite dimensional case: An example

From the Lagrangian

$$L\left(\{x(t), v(t)\}, q(t), \{\lambda_x(t), \lambda_v(t)\}\right)$$
$$= \frac{1}{2}\int_0^T \left(x(t) - (1+t^2)\right)^2 dt - \left\langle \lambda_x, \dot{x}(t) - v(t) \right\rangle - \left\langle \lambda_v, \dot{v}(t) - [-1+q(t)] \right\rangle$$
$$- \lambda_x(0)\left[x(0) - 1\right] - \lambda_v(0)\left[v(0) - 0\right]$$

we get the optimality conditions:

- Derivative with respect to the control function *q(t)*:

$$\int_0^T \lambda_v(t)\rho(t)\,dt = 0 \qquad \forall\, \rho(t)$$

This is equivalent to

$$\lambda_v(t) = 0$$

The infinite dimensional case: An example

The complete set of optimality conditions is now as follows:

State equations:
(initial value problem)

$$\dot{x}(t) - v(t) = 0 \qquad x(0) - 1 = 0$$

$$\dot{v}(t) - (-1 + q(t)) = 0 \qquad v(0) = 0$$

Adjoint equations:
(final value problem)

$$\left(x(t) - (1 + t^2)\right) + \dot{\lambda}_x(t) = 0 \qquad \lambda_x(T) = 0$$

$$\lambda_x(t) + \dot{\lambda}_v(t) = 0 \qquad \lambda_v(T) = 0$$

Control equation:
(algebraic, time dependent)

$$\lambda_v(t) = 0$$

The infinite dimensional case: An example

Let us use all these equations in turn:

Control equation: $\lambda_v(t)=0$

Adjoint equations: $\left(x(t)-(1+t^2)\right)+\dot{\lambda}_x(t)=0 \qquad \lambda_x(T)=0$

$$\lambda_x(t)+\dot{\lambda}_v(t)=0 \qquad \lambda_v(T)=0$$

Solution:

$$\lambda_v(t)=0$$
$$\lambda_x(t)=0$$
$$x(t)=1+t^2$$

Remark: This already implies that we can follow the desired trajectory *exactly*!

The infinite dimensional case: An example

Let us use all these equations in turn:

Now known: $x(t)=1+t^2$

State equation: $\dot{x}(t)-v(t)=0$ $\quad x(0)-1=0$

$\dot{v}(t)-(-1+q(t))=0$ $\quad v(0)=0$

Solution:

$$v(t)=2\mathrm{t}$$
$$q(t)=\dot{v}(t)+1=2+1=3$$

Conclusion: We need a vertical thrust of 3 to offset gravity and achieve the desired trajectory!

Part 28

Optimal control with equality constraints: Theory

Equality constrained optimal control problems

Previously: So far, we have considered optimal control problems where the only constraints were the ODE and initial conditions.

Now: Consider a problem where we also have equality constraints on the state. Specifically, consider final time constraints:

$$
\begin{aligned}
&\min_{x(t)\in X,\, q(t)\in L^2([0,T])} && f(x(t),t,q(t)) = \int_0^T F(x(t),t,q(t))\,dt \\
&\text{such that} && \dot{x}(t) - g(x(t),t,q(t)) = 0 \\
& && x(0) = x_0(q(0)) \\
& && \psi(x(T),q(T),T) = 0
\end{aligned}
$$

Constraints of this form typically occur if we want to be in a certain state (e.g. location) at the end time and seek the minimal energy/ minimal cost path to get there.

Equality constrained optimal control problems

Consider a problem where we also have equality constraints on the state. Specifically, consider final time constraints:

$$\begin{aligned}
&\min_{x(t)\in X, q(t)\in L^2([0,T])} && f(x(t),t,q(t)) = \int_0^T F(x(t),t,q(t))\,dt \\
&\text{such that} && \dot{x}(t) - g(x(t),t,q(t)) = 0 \\
& && x(0) = x_0(q(0)) \\
& && \psi(x(T),q(T),T) = 0
\end{aligned}$$

Then:

$$\begin{aligned}
L(x(t),q(t),\lambda(t),\nu) = \int_0^T F(x(t),t,q(t))\,dt - \langle \lambda, \dot{x}(t) - g(x(t),t,q(t)) \rangle \\
- \lambda(0)[x(0) - x_0(q(0))] - \nu\,\psi(x(T),q(T),T)
\end{aligned}$$

$$L: X \times L^2([0,T]) \times V' \times \mathbb{R} \to \mathbb{R}$$

Equality constrained optimal control problems

Theorem: Under certain conditions the solution satisfies

$$\left\langle \nabla_x L(x^*, q^*, \lambda^*, \nu^*), \xi \right\rangle = 0, \qquad \forall \xi \in X$$
$$\left\langle \nabla_\lambda L(x^*, q^*, \lambda^*, \nu^*), \eta \right\rangle = 0, \qquad \forall \eta \in V$$
$$\left\langle \nabla_q L(x^*, q^*, \lambda^*, \nu^*), \rho \right\rangle = 0, \qquad \forall \rho \in L^2([0,T])' = L^2([0,T])$$
$$\left\langle \nabla_\nu L(x^*, q^*, \lambda^*, \nu^*), \mu \right\rangle = 0, \qquad \forall \mu \in \mathbb{R}$$

Note 1: The last of these equations is simply

$$\psi(x(T), q(T), T) = 0$$

Note 2: The first equation is now

$$\int_0^T F_x(x(t), t, q(t))\xi(t)dt - \left\langle \lambda, \dot{\xi}(t) - g_x(x(t), t, q(t))\xi(t) \right\rangle$$
$$- \lambda(0)\xi(0) - \nu\,\psi_x(x(T), q(T), T)\xi(T) = 0 \quad \forall \xi(t)$$

Equality constrained optimal control problems

Corollary: Given the form of the Lagrangian,

$$L(x(t), q(t), \lambda(t), \nu) = \int_0^T F(x(t), t, q(t))\,dt - \langle \lambda, \dot{x}(t) - g(x(t), t, q(t)) \rangle$$
$$- \lambda(0)[x(0) - x_0(q(0))] - \nu\psi(x(T), q(T), T)$$

the optimality conditions are equivalent to the following four sets of equations:

$$\dot{x}(t) = g(x(t), t, q(t)), \qquad x(0) = x_0(q(0))$$

$$\dot{\lambda}(t) = -F_x(x(t), t, q(t)) - g_x(x(t), t, q(t)), \qquad \lambda(T) = -\nu\psi_x(x(T), q(T), T)$$

$$F_q(x(t), t, q) + \lambda(t) g_q(x(t), t, q) = 0, \qquad \lambda(0)\frac{\partial x_0(q)}{\partial q} = \nu\psi_q(x(T), q(T), T)$$

$$\psi(x(T), q(T), T) = 0$$

These are now called *state equations*, *adjoint equation*, *control equation*, and *transversality equation.*

Equality constrained optimal control problems

Example ("geodesics"): Consider a Mars rover. Given a force vector *q(t)* then it will move with a velocity

$$\dot{x}(t) = \phi(x(t))\, q(t)$$

where the function $\phi(x)$ indicates how "rough/smooth" the terrain is at position *x*: if the terrain is smooth, then $\phi(x)$ is large; if it is rough, then $\phi(x)$ is small.

The goal is then to find a path from x_A to x_B with minimal energy. Let's assume that the power necessary to create a force *q(t)* is equal to $|q(t)|^2$. Then the problem is:

$$\begin{array}{ll} \min_{x(t)\in X,\, q(t)\in L^2([0,T])} & \frac{1}{2}\int_0^T |q(t)|^2\, dt \\ \text{such that} & \dot{x}(t)=\phi(x(t))q(t) \\ & x(0)=x_A \\ & x(T)=x_B \end{array}$$

Equality constrained optimal control problems

Example ("geodesics"): For the problem

$$\min_{x(t)\in X,\, q(t)\in L^2([0,T])} \quad \frac{1}{2}\int_0^T \left(q(t)\right)^2 dt$$

$$\text{such that} \quad \dot{x}(t) = \phi(x(t))\, q(t)$$

$$x(0) = x_A$$

$$x(T) = x_B$$

the Lagrangian is given by

$$L(x(t), q(t), \lambda(t), \nu) = \frac{1}{2}\int_0^T \left(q(t)\right)^2 dt - \left\langle \lambda, \dot{x}(t) - \phi(x(t))\, q(t) \right\rangle - \lambda(0)\left[x(0) - x_A\right] - \nu\left[x(T) - x_B\right]$$

Equality constrained optimal control problems

Example ("geodesics"): The Lagrangian is given by

$$L(x(t),q(t),\lambda(t),\nu) = \frac{1}{2}\int_0^T \left|q(t)\right|^2 dt - \left\langle \lambda, \dot{x}(t) - \phi(x(t))q(t) \right\rangle$$
$$-\lambda(0)[x(0)-x_A] - \nu[x(T)-x_B]$$

The optimality conditions are then:

$$\dot{\lambda}(t) + \nabla\phi(x(t))[q(t)\cdot\lambda(t)] = 0 \qquad \lambda(T) = -\nu$$

$$\dot{x}(t) - \phi(x(t))q(t) = 0 \qquad x(0) = x_A$$

$$q(t) + \lambda\phi(x(t)) = 0$$

$$x(T) = x_B$$

In general, there is no trivial solution to this system.

Equality constrained optimal control problems

Example ("geodesics"): Consider the simplest case, $\phi(x)=1$
Then the optimality conditions are:

$$\dot{\lambda}(t)=0 \qquad \lambda(T)=-\nu$$

$$\dot{x}(t)-q(t)=0 \qquad x(0)=x_A$$

$$q(t)+\lambda=0$$

$$x(T)=x_B$$

This system is solved by

$$\lambda(t)=-\nu \qquad q(t)=\nu$$

$$x(t)=\nu\, t+x_A \qquad \nu=(x_B-x_A)/T$$

That is, the rover moves at constant speed on a straight line and the optimal value of the objective function is $\frac{1}{2}\nu^2 T=\frac{1}{2}\|x_B-x_A\|^2/T$

Equality constrained optimal control problems

Example ("geodesics"): Consider the more difficult case where the rover can move twice as fast in the lower half plane than in the upper half plane:

$$\phi(x) = \begin{cases} 1 & \text{if } x_2 > 0 \\ 2 & \text{if } x_2 \leq 0 \end{cases} = 2 - H(x_2)$$

with $H(y)$=0 for $y=0$. Let $x_A=(-2,1)^T$, $x_B=(2,1)^T$.

Then the optimality conditions are:

$$\dot{\lambda}(t) - \begin{pmatrix} 0 \\ \delta(x_2(t)) \end{pmatrix} [q(t) \cdot \lambda(t)] = 0 \qquad \lambda(T) = -\nu$$

$$\dot{x}(t) - (2 - H(x_2(t)))\, q(t) = 0 \qquad x(0) = x_A$$

$$q(t) + (2 - H(x_2(t)))\, \lambda = 0$$

$$x(T) = x_B$$

Equality constrained optimal control problems

Example ("geodesics"): Consider this difficult case. Equations

$$\dot{\lambda}(t) - \begin{pmatrix} 0 \\ \delta(x_2(t)) \end{pmatrix} [q(t)\cdot\lambda(t)] = 0 \qquad \lambda(T) = -\nu$$

$$\dot{x}(t) - (2 - H(x_2(t)))\,q(t) = 0 \qquad x(0) = x_A$$

$$q(t) + (2 - H(x_2(t)))\,\lambda = 0$$

$$x(T) = x_B$$

have the following solution (note: path is in the upper half):

$$\lambda(t) = -\nu \qquad q(t) = \nu$$

$$x(t) = \nu\, t + x_A \qquad \nu = \frac{1}{T}\begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

The optimal objective function value is then

$$\frac{1}{2}\nu^2 T = \frac{8}{T}$$

Equality constrained optimal control problems

But careful: The conditions

$$\dot{\lambda}(t) - \begin{pmatrix} 0 \\ \delta(x_2(t)) \end{pmatrix} [q(t) \cdot \lambda(t)] = 0 \qquad \lambda(T) = -\nu$$

$$\dot{x}(t) - (2 - H(x_2(t)))\, q(t) = 0 \qquad x(0) = x_A$$

$$q(t) + (2 - H(x_2(t)))\, \lambda = 0$$

$$x(T) = x_B$$

also have a solution of the form

$$x(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -\alpha \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$q(t) = \text{const}$$

(Details to be determined. We also have to specify in more detail what it means if we move along the line $x_2 = 0$, c.f. the first equation above.)

Part 29

Direct vs. indirect methods

Direct vs. indirect methods

How do we solve general optimal control problems:

- *Direct methods* are based on the original problem formulation.
  We can think of them as "discretize first, then optimize".
- *Indirect methods* attempt to solve the optimality conditions.
  We can think of them as "optimize first, then discretize"

Example: To find a minimum of $f(x)$,

- Direct methods would find a sequence x_1, x_2, … and would only have to ensure that $f(x_1) > f(x_2), \ldots$

  I.e. it would only have to *compare* function values.

- Indirect methods would try to find a solution of the equation $f'(x)=0$.

  I.e. we would have to compute *derivatives* of the objective function.

Direct vs. indirect methods

In practice, all methods in actual use are *direct*:

- For many realistic problems, the user-defined function $F, g, \ldots$ are complicated and providing derivatives for the necessary conditions is not practical
- Good initial estimates for the Lagrange multipliers are typically not available
- Without good initial estimates, indirect methods often wander off into lala-land unless the problem is exceptionally stable
- With state inequalities, we need to provide an a-priori guess when the inequalities will be active. This is not practical.

Consequently: The optimality conditions derived so far are of mostly theoretical interest in optimal control. They are of importance in PDE-constrained optimization, however.

Part 30

Numerical solution of optimal control problems with direct methods

The shooting method for realistic optimal control

Consider a problem with equality constraints on the state: Specifically, consider final time constraints:

$$
\begin{aligned}
&\min_{x(t)\in X,\, q(t)\in L^2([0,T])} && f(x(t),t,q(t)) = \int_0^T F(x(t),t,q(t))\,dt \\
&\text{such that} && \dot{x}(t) - g(x(t),t,q(t)) = 0 \\
& && x(0) = x_0(q(0)) \\
& && \psi(x(T),q(T),T) = 0
\end{aligned}
$$

Approach: We want to apply a (single) shooting method to it. To this end, introduce a time mesh

$$0 = t_0 < t_1 < t_2 < \ldots < t_N = T$$

and a time step size $k_n = t_n - t_{n-1}$.

We then apply one of the common time stepping methods to the optimal control problem. (This step is called “discretization”.)

The shooting method for realistic optimal control

Consider a problem with equality constraints on the state: Specifically, consider final time constraints:

$$\begin{aligned}
&\min_{x(t)\in X,\, q(t)\in L^2([0,T])} && f(x(t),t,q(t))=\int_0^T F(x(t),t,q(t))\,dt \\
&\text{such that} && \dot{x}(t)-g(x(t),t,q(t))=0 \\
& && x(0)=x_0(q(0)) \\
& && \psi(x(T),q(T),T)=0
\end{aligned}$$

Example: Using the (overly trivial, low-order) forward Euler method, replace the original problem with the discretized form

$$\begin{aligned}
&\min_{x^n,q^n,\, n=0,\dots,N} && f(x^0,\dots,x^N,q^0,\dots,q^N)=\sum_{n=1}^N k_n F\left(\frac{x^n+x^{n-1}}{2},t_n,\frac{q^n+q^{n-1}}{2}\right) \\
&\text{such that} && \frac{x^n-x^{n-1}}{k_n}-g(x^{n-1}(t),t_{n-1},q^{n-1}(t))=0 \\
& && x^0=x_0(q^0) \\
& && \psi(x^N,q^N,T)=0
\end{aligned}$$

The shooting method for realistic optimal control

The discretized problem now reads as:

$$\min_{x^n, q^n,\, n=0,\dots,N} \quad f(x^0,\dots,x^N,q^0,\dots,q^N) = \sum_{n=1}^{N} k_n F\left(\frac{x^n+x^{n-1}}{2}, t_n, \frac{q^n+q^{n-1}}{2}\right)$$

$$\text{such that} \qquad \frac{x^n-x^{n-1}}{k_n} - g(x^{n-1}(t), t_{n-1}, q^{n-1}(t)) = 0$$

$$x^0 = x_0(q^0)$$

$$\psi(x^N, q^N, T) = 0$$

Note: Introducing $y=(x^0,q^0,x^1,q^1,\dots x^N,q^N)^T$ this has the form

$$\begin{aligned} \min_y \quad & f(y) \\ \text{such that} \quad & c(y) = 0 \end{aligned}$$

If *x(t)* has n_x components and *q(t)* has n_q components, then

$$y \in \mathbb{R}^{(N+1)(n_x+n_q)}, \qquad c \in \mathbb{R}^{Nn_x+n_x+n_\psi}$$

The shooting method for realistic optimal control

The discretized problem is now equivalent to a large, nonlinear optimization problem:

$$\begin{array}{ll} \min_y & f(y) \\ \text{such that} & c(y)=0 \end{array}$$

Its solution has to satisfy

$$\begin{aligned} \frac{\partial L}{\partial y} &= \frac{\partial f(y)}{\partial y} - \lambda^T \frac{\partial c(y)}{\partial y} = 0 \\ \frac{\partial L}{\partial \lambda} &= c(y) = 0 \end{aligned}$$

where $L(y,\lambda)=f(y)-\lambda^T c(y)$.

Note: We have one Lagrange multiplier for each time step, but these are all independent. Conversely, in the indirect approach, we would have had Lagrange multipliers for each time step that satisfy a discrete ODE and are therefore all coupled.

 This is what makes the direct method more practical.

The shooting method for realistic optimal control

We can solve this problem using, for example, the SQP method:

$$\begin{pmatrix} \nabla_y^2 f(y_k) - \lambda_k^T \nabla_y^2 c(y_k) & -\nabla_y c(y_k) \\ -\nabla_y c(y_k)^T & 0 \end{pmatrix} \begin{pmatrix} p_k^y \\ p_k^\lambda \end{pmatrix} =$$

$$= -\begin{pmatrix} \nabla_y f(y_k) - \lambda_k^T \nabla_y c(y_k) \\ -g(y_k) \end{pmatrix}$$

We will abbreviate this as

$$\begin{pmatrix} W_k & -A_k \\ -A_k^T & 0 \end{pmatrix} \begin{pmatrix} p_k^y \\ p_k^\lambda \end{pmatrix} = -\begin{pmatrix} \nabla_y f(y_k) - \lambda_k^T \nabla c(y_k) \\ -c(y_k) \end{pmatrix}$$

where

$$W_k = \nabla_y^2 L(y_k, \lambda_k)$$

$$A_k = \nabla_y c(y_k) = -\nabla_x \nabla_\lambda L(y_k, \lambda_k)$$

The shooting method for realistic optimal control

In each iteration, we have to solve the linear system

$$\begin{pmatrix} W_k & -A_k \\ -A_k^T & 0 \end{pmatrix} \begin{pmatrix} p_k^y \\ p_k^\lambda \end{pmatrix} = -\begin{pmatrix} \nabla_y f(y_k) - \lambda_k^T \nabla c(y_k) \\ -c(y_k) \end{pmatrix}$$

The matrix on the left has dimensions

$$\begin{aligned} &\left[(N+1)(n_x+n_q)+Nn_x+n_x+n_\psi\right] \times \left[(N+1)(n_x+n_q)+Nn_x+n_x+n_\psi\right] \\ &= \left[(N+1)(n_x+1+n_q)+n_\psi\right] \times \left[(N+1)(n_x+1+n_q)+n_\psi\right] \end{aligned}$$

Note: It is not uncommon to have 10-100 state variables, 1-10 control variables, and 1,000-10,000 time steps. That means the matrix on the left can easily be of size $10{,}000^2$ to $1{,}000{,}000^2$!

That would be a very large and awkward system to solve in each iteration!

The shooting method for realistic optimal control

Conclusion so far: The SQP system

$$\begin{pmatrix} W_k & -A_k \\ -A_k^T & 0 \end{pmatrix} \begin{pmatrix} p_k^y \\ p_k^\lambda \end{pmatrix} = -\begin{pmatrix} \nabla_y f(y_k) - \lambda_k^T \nabla c(y_k) \\ -c(y_k) \end{pmatrix}$$

is very large.

However: The matrix on the left is also almost completely empty. Remember that

$$W_k = \nabla_y^2 L(y_k, \lambda_k) = \nabla_y^2 f(y_k) - \sum_i (\lambda_{k,i}) \nabla_y^2 c_i(y_k)$$

$$A_k = \nabla_y c(y_k)$$

and that

$$f(y) = \sum_{n=1}^N k_n F\left(\frac{x^n + x^{n-1}}{2}, t_n, \frac{q^n + q^{n-1}}{2}\right)$$

$$c(y) = \begin{pmatrix} \frac{x^n - x^{n-1}}{k_n} - g(x^{n-1}(t), t_{n-1}, q^{n-1}(t)) \\ x^0 - x_0(q^0) \\ \psi(x^N, q^N, T) \end{pmatrix}$$

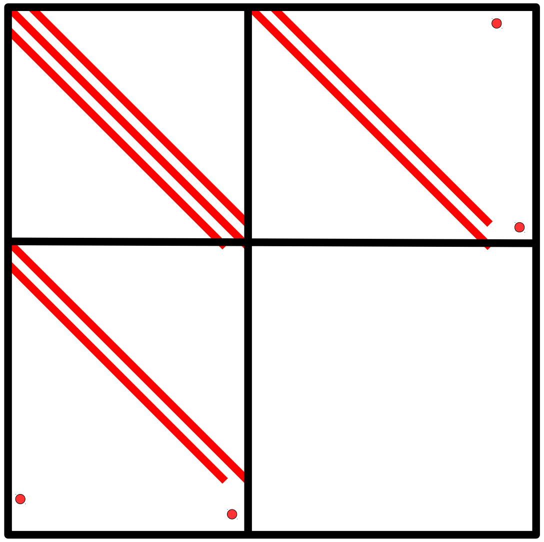
The shooting method for realistic optimal control

Conclusion so far: The SQP system

$$\begin{pmatrix} W_k & -A_k \\ -A_k^T & 0 \end{pmatrix} \begin{pmatrix} p_k^y \\ p_k^\lambda \end{pmatrix} = -\begin{pmatrix} \nabla_y f(y_k) - \lambda_k^T \nabla c(y_k) \\ -c(y_k) \end{pmatrix}$$

is very large.

However: The matrix on the left is also almost completely empty. It typically has a (block) structure of the form



Note: Such systems are not overly complicated to solve.

The multiple shooting method

Instead of using the single shooting method,

$$\min_{x^n, q^n, n=0,\ldots,N} \; f(x^0,\ldots,x^N,q^0,\ldots,q^N) = \sum_{n=1}^{N} k_n F\left(\frac{x^n+x^{n-1}}{2}, t_n, \frac{q^n+q^{n-1}}{2}\right)$$

such that

$$\frac{x^n - x^{n-1}}{k_n} - g(x^{n-1}(t), t_{n-1}, q^{n-1}(t)) = 0$$

$$x^0 = x_0(q^0)$$

$$\psi(x^N, q^N, T) = 0$$

we relax the formulation to obtain the multiple shooting method:

$$\min_{x^{s,n}, q^{s,n}, n=0,\ldots,N_s, s=1\ldots S} \; \sum_{s=1}^{S} \sum_{n=1}^{N_s} k_{s,n} F\left(\frac{x^{s,n}+x^{s,n-1}}{2}, t_{s,n}, \frac{q^{s,n}+q^{s,n-1}}{2}\right)$$

such that

$$\frac{x^{s,n} - x^{s,n-1}}{k_{s,n}} - g(x^{s,n-1}(t), t_{s,n-1}, q^{s,n-1}(t)) = 0, \quad s=2\ldots S$$

$$x^{1,0} = x_0(q^{1,0})$$

$$x^{s,0} = x^{s-1,N_{s-1}}, \qquad s=2\ldots S$$

$$\psi(x^{S,N_S}, q^{S,N_S}, T) = 0$$

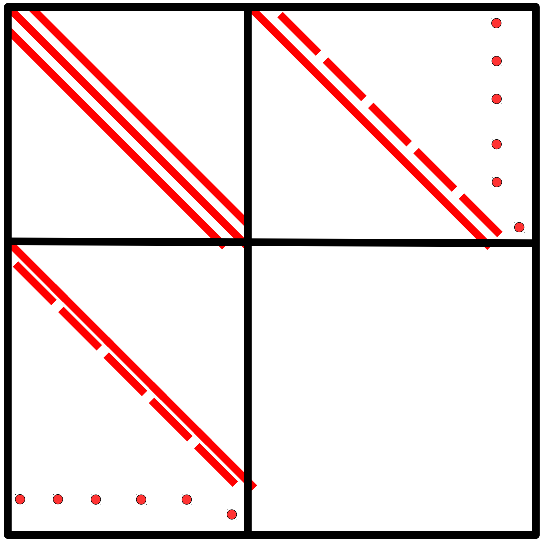
The multiple shooting method

Multiple shooting method: The SQP system has the form

$$\begin{pmatrix} W_k & -A_k \\ -A_k^T & 0 \end{pmatrix} \begin{pmatrix} p_k^y \\ p_k^\lambda \end{pmatrix} = -\begin{pmatrix} \nabla_y f(y_k) - \lambda_k^T \nabla c(y_k) \\ -c(y_k) \end{pmatrix}$$

with now even more variables.

However: The matrix on the left is again also almost completely empty. It typically has a (block) structure of the form



Note: Again, such systems are not overly complicated to solve. In particular, this system can now also be solved in parallel.

Time stepping vs. SQP

Remark: A typical strategy of coupling time discretization and nonlinear optimization is

- to start with a relatively small number of time steps
- do one or more SQP steps
- interpolate the current solution variables x^n, q^n as well as the Lagrange multipliers to a finer time mesh
- do some more SQP iterations and iterate this procedure

Advantages:

- While we are far away from the solution, the number of variables is small and so every SQP step is fast
- Only close to the solution do iterations get expensive
- The degree of ill-posedness of problems typically increases with smaller time steps. We can work with well-posed problems while we need to take large steps, stabilizing the process.