MATH 437: Principles of Numerical Analysis

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Homework assignment 9 – due Thursday 11/7/2013

Problem 1 (Best polynomial approximation). Compute, analytically (i.e. with exact values, not numerical floating point values), the best (least squares) polynomial approximation of degree 4 on the interval $[-1, 1]$ to the following functions:

a) $f(x) = \frac{1}{x}$;

a) $f(x) = e^x$.

Plot your best approximation $p_4(x)$ together with $f(x)$. **(5 points)**

Problem 2 (Least-squares approximation and other norms). In class, we defined the least-square approximating polynomial $p_q(x) = \sum_{k=0}^{q} a_k x^k$ as that polynomial that minimized the error

$$e_2(a_0, \dots, a_q) = \sum_{i=1}^{N} |p_q(x_i) - y_i|^2,$$

where we used the l_2 norm of the difference between $p_q(x_i)$ and y_i (i.e., we squared the difference, and summed over it). We convinced ourselves that this then leads to a linear problem for finding the expansion coefficients a_k.

On the other hand, we could choose any other norm as well to define the error. For example, we could have chosen the l_4 norm

$$e_4(a_0, \dots, a_q) = \sum_{i=1}^{N} |p_q(x_i) - y_i|^4,$$

or the l_∞ norm

$$e_\infty(a_0, \dots, a_q) = \max_{i=1}^{N} |p_q(x_i) - y_i|,$$

and finding the coefficients a_k that minimize one of these functions will, most likely, yield a different polynomial $p_q(x)$ than the the one from before. Both of these polynomials are optimal, but with regard to different notions of the difference between $p_q(x_i)$ and y_i. It is relatively easy to see that if we choose any other norm than the l_2 norm, then the problem of finding the a_k is a nonlinear one.

Take the same points from last week again:

x_i	1	2	3	4	5	6	7	8	9	10
y_i	1.51	2.01	2.49	2.98	3.51	4.01	4.49	5.02	5.52	5.98

Let us try to fit a linear function $p_1(x) = c_0 + c_1 x$ to these points using the different notions of "error". Do the following:

- For $r \in \{1, 2, 4, \infty\}$, plot the functions $e_r(c_0, c_1)$ as surfaces over the axes c_0, c_1. Determine whether these functions e_r are smooth and whether, consequently, finding a minimum can be done by just setting the derivative to zero.

- For each of these values r, find the polynomial $p_q(x)$ that minimizes the error e_r. (For $r = 2$, this is the solution of Problem 4 of last week's homework.) Plot all of these polynomials together in one plot in which you also show the 10 data points.

Repeat these computations for the following data set (the third to last data point has been changed: some large measurement error has occured, or someone made a mistake transfering the device reading to the data sheet; or maybe this was what the experiment really gave):

x_i	1	2	3	4	5	6	7	8	9	10
y_i	1.51	2.01	2.49	2.98	3.51	4.01	4.49	5.82	5.52	5.98

Comment on the suitability of the solutions you've found for approximating the two data sets.

Note: To compute each of these polynomials, you have to find the coefficients c_0, c_1 that minimize the respective error $e_r(c_0, c_1)$. In general, you will not be able to find these coefficients exactly except for the case $r = 2$. In this case, feel free to get approximate values of the coefficients by plotting e_r as described above and visually determining values for which it is minimal. **(8 points)**