MATH 437: Principles of Numerical Analysis

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Homework assignment 3 – due Thursday 9/19/2013

Problem 1 (Gaussian elimination). Solve (on paper, showing the individual steps) the following system of linear equations using Gaussian elimination:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Verify that your result is correct.

(The matrix in this example is the so-called Hilbert matrix, with entries $H_{ij} = \frac{1}{i+j-1}$. It has a number of nasty properties that make it a good testcase for matrix algorithms, see http://en.wikipedia.org/wiki/Hilbert_matrix.)

(4 points)

Problem 2 (Gaussian elimination). Write a computer function that takes a general $n \times n$ matrix A as input and computes its inverse A^{-1} as output. Implement the algorithm by hand, i.e., you shouldn't just call the Matlab function that computes the inverse.

Apply this function to compute, numerically, the solution of Problem 1. Do the same for Hilbert matrices of sizes 10×10, 100×100 and 200×200. In each case, verify numerically that $AA^{-1} = I$ where I is the identity matrix. What do you observe? **(6 points)**

Problem 3 (Gaussian elimination). Using Gaussian elimination, it is simple to solve the following problem

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

One would eliminate the occurrence of x_1 in the second equation by subtracting the first from the second equation, arriving at a diagonal matrix.

Describe what happens if the system instead looked like this:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Does the algorithm still work? If not, propose a remedy. **(2 points)**

Problem 4 (Multidimensional root finding). Use Newton's method and your function from Problem 2 to find the minimum of

a) $g(x, y) = e^x - \cos(x + y) - \cos(x - 2y)$,

b) $g(x, y) = e^{y - \sin(4x)} + e^{-y + \sin(4x)}$,

in both cases starting at $x_0 = y_0 = 1$. Remember that finding the minimum of $g(x, y)$ amounts to finding a simultaneous root of the system

$$f_1(x, y) = \frac{\partial g(x, y)}{\partial x} = 0,$$
$$f_2(x, y) = \frac{\partial g(x, y)}{\partial y} = 0.$$

What happens if you start at $x_0 = y_0 = 0$ in case b)? **(5 points)**