MATH 437: Principles of Numerical Analysis

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Homework assignment 2 – due Thursday 9/12/2013

Problem 1 (Taylor series). Many important functions such as the sine cannot be computed in a simple way, i.e. with only the four basic operations plus, minus, multiplication and division. However, they can be approximated with these operations.

Graph the first eight Taylor approximations of $f(x) = \sin x$ when expanded around zero, i.e.

$$
\begin{aligned}
f_1(x) &= f(0) + f'(0)x, \\
f_2(x) &= f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2, \\
f_3(x) &= f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + \tfrac{1}{3!}f'''(0)x^3,
\end{aligned}
$$

etc. What do you observe? What does this mean for the approximation of $f(2\pi)$?

Write a program to experimentally determine the number of terms you need to approximate $f(2\pi) = 0$ to an accuracy of 10^{-4} and 10^{-12}. **(6 points)**

Problem 2 (Computing roots by primitive operations) In Problem 1, you saw how a Taylor series can be used to compute the value of a non-primitive function (i.e. a function that does not only require the four basic operations) can be approximated using only primitive operations.

Consider the non-primitive function $g(x) = \sqrt{x}$. Compare the following two methods to compute $g(2) = \sqrt{2}$:

a) Derive a Taylor series approximation around $x_0 = 1$ to compute $g(2)$.

b) A Newton iteration starting at $x_0 = 1$ to find the root of the function $f(x) = x^2 - 2$.

Write two small programs that compute $f(2)$ with these two methods to n digits of accuracy. Provide the number of operations (additions, subtractions, multiplications, divisions) to reach $n = 3, 5, 10$ digits. **(6 points)**

Problem 3 (Newton's method). For certain functions, Newton's method will always converge in a single step, no matter where we start. What functions are these, and why is a single step enough? (Hint: Think about the graphical interpretation of Newton's method, and when it will produce a new iteration that falls exactly onto the true root of the function.) **(2 points)**

Problem 4 (Solving nonlinear equations). Assuming that the Riemann Conjecture is true, it is known that the number $\pi(x)$ of primes in the range $1 \ldots x$ can be approximated by

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2}.$$

(See

`http://en.wikipedia.org/wiki/Riemann_hypothesis`

The formula above contains the first two terms of the Taylor expansion of the integral form of $\pi(x)$ given on that webpage.)

Write a program to compute, using a method of your choice, the number x for which the approximate formula predicts that there are 10^4 prime numbers less than x, i.e. solve the equation

$$\pi(x) = 10^4.$$

To what accuracy does it make sense to solve this problem? **(4 points)**

Problem 5 (Newton's method). For functions $f(x)$ of one variable x, Newton's method almost always converges very quickly (in a matter of a few iterations). However, almost always is not always, and we can find examples where Newton's method converges rather slowly.

Write a program to find the zero $x = 1$ of the function

$$f(x) = x^{25} - 1$$

that uses Newton's method and starts at $x_0 = 20$.

(a) How many iterations do you need to achieve an accuracy of 10^{-8}?

(b) You will observe very slow convergence. Can you explain from the formulas that express the error e_n as a function of e_{n-1} why convergence is so slow?

(c) Does the method still converge of second order, i.e. is the relationship between e_n and e_{n+1} derived in class true also for this problem?

(d) What answers do you get to the questions in (a)–(c) if you apply the same program to the function $f(x) = x^3 - 1$ instead, again starting from $x_0 = 20$? **(8 points)**