MATH 601: Quiz 10 (11/14/2012)

Name: UIN:

Problem 1 (5 points): Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and answer the following questions:

- What is the characteristic polynomial of this matrix?

  Answer: The characteristic polynomial is defined as the determinant of the matrix $A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$. Here, it is equal to

  $$\det(A - \lambda I) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

- What is the relationship between the characteristic polynomial of this matrix and the eigenvalues of the matrix? Based on the degree of the polynomial, how many eigenvalue can at most exist for this matrix?

  Answer: The eigenvalues of a matrix are defined as the roots of the characteristic polynomial, i.e., as those values λ for which the characteristic polynomial is zero. Because the polynomial is quadratic (i.e., of degree 2), there can be at most 2 such roots based on the fundamental theorem of algebra.

- Find the eigenvalues of the matrix. How many distinct ones are there?

  Answer: Based on the factorization above, we then seek λ so that $(\lambda - 3)(\lambda - 1) = 0$. This is clearly the case for $\lambda_1 = 3, \lambda_2 = 1$. Thus, there are two distinct eigenvalues.

- Find the eigenvectors that correspond to each of the eigenvalues. How many such eigenvectors exist?

  Answer: Let us first consider the eigenvector(s) to $\lambda_1 = 3$. We then need to find vectors v so that $Av = 3v$, i.e., vectors so that $(A - 3I)v = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} v = 0$. One vector that clearly satisfies this is $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (Any multiple of this vector of course also satisfies it but we are not interested in distinguishing between multiples of vectors. In other words, we only count v_1 and $3v_1$ as a single eigenvector.)

  We can repeat the same procedure for the second eigenvalues, $\lambda_2 = 1$. There, we need to find v so that $(A - 1I)v = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v = 0$, which is the case for $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as well as all of its multiples.

  In summary, we have two eigenvectors (plus all their multiples).

- Define what it means if a matrix is diagonalizable. Is A diagonalizable?

  Answer: A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if there is a matrix $P \in \mathbb{R}^{n \times n}$ so that $P^{-1}AP$ is diagonal. In particular, we have seen in class that a matrix is diagonalizable if the matrix has n linearly independent eigenvectors. This is the case here (v_1 and v_2 are not linearly dependent) and consequently our matrix A is diagonalizable.

(see backside)

Problem 2 (3 points): Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and answer the following questions:

- Find the eigenvalues of the matrix. (Note the special structure of the matrix to make your life simpler.) How many distinct ones are there?

  Answer: Because the matrix is triangular, its diagonal entries are also its eigenvalues. There are two, $\lambda_1 = 2, \lambda_2 = 2$. (Although there really are two eigenvalues, because they are the same, one may say that there is only a single distinct value.)

- Find the eigenvectors that correspond to each of the eigenvalues. How many such eigenvectors can you find?

  Answer: As for Problem 1 we need to find vectors so that $(A - \lambda I)v = 0$. Here, this means finding vectors so that

  $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v = 0.$$

  Ideally, because there are two eigenvalues, we would like to find two linearly independent such vectors v_1, v_2. However, it is the case here that there is only one such vector, $v_1 = (1, 0)^T$ along with all of its multiples.

- Is A diagonalizable?

  Answer: No, because there is only one eigenvector and not a full complement of $n = 2$ linearly independent eigenvalues.

Problem 3 (2 points): Based on the recursive definition of the determinant of an $n \times n$ matrix, count how many operations (additions and multiplications) you need to compute the determinants of matrices if sizes $2 \times 2, 3 \times 3, 4 \times 4$ and 5×5.

Answer: When forming the determinant of a 2×2 matrix, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ac - bd$, we need 2 multiplications plus one addition, i.e., a total of 3 operations. Let us therefore say that for matrices of size 2 we need $O(2) = 3$ operations.
Next remember that for 3×3 matrices, we compute the determinant as

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - d \det \begin{pmatrix} b & c \\ h & i \end{pmatrix} + g \det \begin{pmatrix} b & c \\ e & f \end{pmatrix}.$$

In other words, to compute this, we need to compute three sub-determinants of size 2 ($3O(2)$ operations), multiply each of them with a number (3 operations) and add them together (2 operations). Thus, $O(3) = 3(O(2) + 1) + 2 = 14$.
Using a similar construction, we find $O(4) = 4(O(3) + 1) + 3 = 63$ and $O(5) = 5(O(4) + 1) + 4 = 324$. The general formula is equally easily derived: $O(n) = n(O(n-1) + 1) + (n-1) = n(O(n-1) + 2) - 1$.
To see how quickly this grows, consider that even for modest matrix sizes, say $n = 5$, we have seen that $O(n)$ is already pretty large. Consequently, for larger n, we have $O(n) \approx nO(n-1) \approx n(n-1)O(n-2) \approx n(n-1)(n-2)O(n-3) = \ldots$. That means that $O(n)$ grows about as rapidly as $n!$ (pronounced n factorial), a number sequence that grows exponentially fast. As we have investigated in class, even computing the determinant of a 20×20 matrix will not be possible using this method, let alone computing it for even larger matrix sizes.
