MATH 442: Mathematical Modeling

Lecturer: Prof. Wolfgang Bangerth
Blocker Bldg., Room 507D
(979) 845 6393
`bangerth@math.tamu.edu`
`http://www.math.tamu.edu/~bangerth`

Individual project – due 12/9/2010

This project is about investigating the spread of pests and diseases. Since this is an individual project, you are supposed to do most of the work yourself and to acknowledge all sources of information (including help you receive from others) appropriately.

Your task consists of producing a report – similar in style to the one for the first project – in which you discuss your results. Your report should include an introduction in which you state the problem including all formulas, a methods section in which you discuss how you approached the problem and how you implemented it, and a results section in which you cover your results comprehensively. You may alternate methods and results sections for Parts 3 and 4. Put particular emphasis not only on stating what you find but also on interpreting the results. Your report should be around 10 pages, including figures. You also need to provide your program code in electronic form to me. Programs may be written in Maple or any other programming environment you feel comfortable with.

Note: You will realize that this assignment has a lot of parts. You are not expected to work on all of them (though of course you can). In fact, the total number of points adds up to more than 100, allowing you to choose those parts that suit your talents.

Pests and chronic diseases

Let's consider things like pests or chronic diseases: these are organisms (let's collectively call them *pathogens*) that infect hosts (e.g. plants or humans) but for which the host's defenses are not strong enough to completely eradicate. This can be seen everywhere: Plants can be infected with aphids; trees by bark beetles; and humans by herpes simplex (HSV), hepatitis C (HCV) or human papilloma viruses (HPV). In all of these cases, infections are chronic. This is opposed to, for example, the common flu that also infects us but after some initial training period the human immune system is able to target flu viruses with sufficient specificity to completely purge them from the body, returning the population of viruses in the body to zero.

For chronic infections, let us consider the following interactions with the hosts:

- In the early stages of an infection, the pathogen population will grow exponentially.
- As the population becomes larger, host defenses become stronger and competition for available resources fiercer, limiting population growth.
- The host will not be able to completely purge the pathogen from its system, leading to a chronic infection: an equilibrium is found between pathogen reproduction and pathogen death due to lack of resources or immune system attacks.
- A small fraction of the pathogen population is able to escape the host and infect other hosts. Infection can only happen if there is an avenue for infection between the two hosts. For example, aphids can infect another plant if it is close enough by riding the wind (or being dispersed by ants); beetles can fly a certain distance; and HSV, HCV or HPV can jump from one host to another through the exchange bodily fluids.
- The number of pathogens that per time unit leaves a host is a constant fraction of the total population on this host. For example, each day 10% of aphids may leave a plant in search for greener pastures.
- The number of pathogens that arrive at host i from host j is a fraction that depends on some distance relationship between hosts i and j. For example, if you have the flu and sneeze (you release, say, 10^9 viruses) in a room in which I am as well, there is an avenue of transmission. If I stand on the other side of the room, I will only inhale 10^3 of them whereas if we stood closer to each other it may be 10^5.

Part 1 (The simplest model). Consider that there are just two hosts and that there is a method of infection between the two of them. Derive a set of differential equations for the number of pathogens $x_1(t), x_2(t)$ in the two hosts that satisfy the general principles above. Describe the individual terms and coefficients in your equations as well as what initial conditions need to be posed to make the model complete. **(10 points)**

Part 2 (A graph based model). Now consider the more complicated case where there are N hosts and we are interested in the number of pathogens $x_i(t), i = 1 \dots N$ for each host. Complete the same tasks as for Part 1. To do so, you should consider the hosts as the vertices of a graph and possibilities for transmission as the edges; the likelihood of transmission (e.g. the distance in the introduction above) would be an edge weight. For the moment you can assume that the edges and edge weights of the graph are given somehow, i.e. use symbols instead of concrete values (we will consider particular graphs below).
(20 points)

Part 3-a (Aphids in the nursery). Let us now consider a nursery in which plants are arranged on a neat x-y-lattice at equal intervals – let's say there are M_x plants in the x-direction at a distance of $\Delta x = 0.3m$ each, and M_y in the y-direction at a distance of $\Delta y = 0.4m$ each, for a total of $N = M_x M_y$. On each plant a population of aphids lives (the population could be zero). Our hypothetical aphids are terrible fliers: they can drift with the wind for a maximal distance of $1.5m$: if they don't land on a plant by then they will fall to the ground and starve; if they do land on a plant they join the local population (or start the population if the number of aphids had been zero before).

Given these flight characteristics, describe in words and formulas the graph that describes this situation, i.e. state what nodes you have and what edges connect them. Visualize this graph for modest numbers M_x, M_y. Make reasonable assumptions (and explain them) for edge weights that describe the likelihood of transmission along an edge. **(15 points)**

Part 3-b (Visualization). Use a computer program to find a graphical representation of this graph for $M_x = M_y = 20$. There are many that you can probably find on the internet for this task. One that is available on the calclab machines is the command line program `neato` (see `http://www.graphviz.org/pdf/neatoguide.pdf`); there are certainly also ways to do this in Maple. The input to any such program is a description of the sets of nodes and edges; the output is an image file. Remember that the graph itself only consists of nodes and edges and has nothing to do with the geometric location of these nodes in the real world. Part of the grading of this problem will be the artistic quality of the image. **(5 points)**

Part 3-c (Numerical solution). Take the model you derived in Part 2 and the graph you got from Part 3-a and implement a program that solves these equations numerically, assuming that initially only a single plant has aphids. Your model likely has several parameters (e.g. the fraction of aphids that leave their plant's population per unit of time; the fraction of aphids that actually arrive at a destination; initial values on the single infested plant; etc.); choose reasonable values for this system and describe how you arrived at them. Document the numerical solution by plotting the aphid population on a variety of plants throughout the nursery and interpret any patterns you may see. Compare your results with what you would intuitively expect to see.

(15 points)

Part 3-d (Visualization). Try to visualize the aphid population on all plants at once at various times of your simulation. Alternatively you could produce a movie that shows populations. Again, part of the grading of this problem will be the artistic quality. Interpret your results. **(10 points)**

Part 4-a (Chronic diseases on random graphs). Now consider chronic diseases and how they spread among humans. The model will be essentially the same as in Part 2 but since humans are not stationary like plants, they prove to be far better vehicles for the spread of diseases (witness, for example, the spread of the SARS virus). On the other hand, the graphs on which diseases spread are also more complicated. In particular, it is much more time consuming to determine such graphs in real world because it requires interviewing actual people and investigating their habits.

As a consequence, the spread of diseases is often investigated on artificially generated graphs. (In research, these graphs often have millions or more vertices, but we won't go this far for your project.) A popular approach to this end are *scale-free networks* (a special case of random graphs). Find appropriate literature that describes what these networks are and how one can construct such randomly generated graphs; summarize your findings in your own words. State how the model derived in Part 2 will now look and explain how you would expect a disease to spread from a single infected person; compare your expectations with those for the aphid model. **(15 points)**

Part 4-b (Sample graphs). Use one of the methods described in the literature to generate a scale-free network with 400 vertices, and one with 10,000 vertices. Visualize these graphs. (Note: Visualizing graphs with 10,000 vertices is not difficult, but it is difficult to do in a way that makes it easy to understand what is happening. The goal is not just to produce *a* picture, but a picture that leads to insight.) **(15 points)**

Part 4-c (Numerical solution). Take the model you derived in Part 2 and one of the graphs you got from Part 4-b and implement a program that solves these equations numerically, assuming that initially only a single person is affected. As in the aphid model, choose and document reasonable coefficients for your model. Document the numerical solution by plotting the pathogen population in several people and interpret any patterns you may see. Compare your results with what you would intuitively expect to see. **(15 points)**

Part 4-d (Visualization). Try to visualize the pathogen populations in all humans at once. Again, part of the grading of this problem will be the artistic quality of the image. Interpret your results. **(20 points)**

I try to be as good a teacher as possible, but to succeed in this goal I need feedback from those who see me teach, i.e. you. If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, group work vs. whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!