MATH 651: Optimization I

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Homework assignment 5 – due Thursday 10/15/2009

Problem 1 (A nontrivial example, revisited). Last week's problem 2 didn't really provide much in terms of complexity. Try again with this function $f : \mathbb{R}^N \to \mathbb{R}$:

$$f(x) = \sum_{i=1}^{N} (x_i - i)^2 + \sum_{i=2}^{N} \left[i(x_i - x_{i-1}^2)^2 \right],$$

with $N = 10$. Show convergence by choosing randomly 100 starting points of the form $x_0 = (\pm 1, \pm 1, \dots, \pm 1)^T$ and determining where your iteration converges. Verify with the necessary conditions that the points your algorithms converges to are indeed minima.

This is a challenging problem. Unless you are absolutely certain that you've found the minimum, provide your source code to get at least partial credit.

(6 points)

Can you use your program to also find the minimum of $f(x)$ if $N = 50$? $N = 100$? $N = 1000$? **(up to 2 bonus points)**

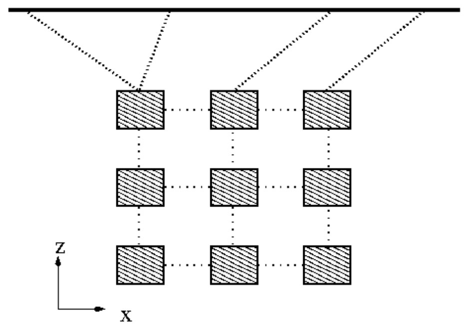
Problem 2 (BFGS for quadratic functions?). Consider $f : \mathbb{R}^4 \to \mathbb{R}$ given by $f(x) = \frac{1}{2} x^T A x$ with

$$A = \begin{pmatrix} 9 & 1 & 1 & 1 \\ 1 & 9 & 1 & 1 \\ 1 & 1 & 9 & 1 \\ 1 & 1 & 1 & 9 \end{pmatrix}$$

Its minimum lies at $x = 0$. Implement the BFGS algorithm for this problem, using full step length $\alpha_k = 1$ in each step, starting at $x_0 = (1, 2, 3, 4)^T$. How many steps do you need to converge to the minimum? How many steps would you need if you used the exact Newton matrix instead of the BFGS approximation?

(4 points)

Problem 3 (Not just three springs). While at it, let's also revisit another one of last week's problems. Consider this system here:



The four springs at the top are suspended from the ceiling at positions $(x, z) = (-20\text{cm}, 0\text{cm})$, $(-5\text{cm}, 0\text{cm})$, $(7\text{cm}, 0\text{cm})$ and $(15\text{cm}, 0\text{cm})$ and have rest lengths of $L_0^{\text{top}} = 20\text{cm}$. The 9 bodies are connected with 12 springs of rest lengths $L_0^{\text{between}} = 5\text{cm}$. All springs are attached to the centers of the sides of the bodies, which all have a size of $2\text{cm} \times 2\text{cm}$ and have a mass of $m = 500g$.

Express the total energy in the system (spring energies plus potential energies) as a function of the 9 bodies' positions $(x_i, z_i), i = 1, \ldots, 9$. Nature still likes to do things so that the energy is minimal, so use a line search quasi-Newton method to find the location at which this energy is minimal. Use the BFGS formula to approximate the inverse of the Hessian matrix and compute quasi-Newton updates with it. (Note that this saves you the trouble of computing awkward looking second derivatives of the objective function, and you also don't have to worry whether the Hessian may have negative eigenvalues.)

In your answer, give the location of the centers of the 9 bodies as well as the value of the total energy function at the minimum. **(6 points)**

Problem 4 (Is the BFGS matrix always positive definite?). Given a previous BFGS matrix B_k, and given previous steps $s_k = x_k - x_{k-1}$ and changes in the gradient $y_k = \nabla f(x_k) - \nabla f(x_{k-1})$, the BFGS method computes the next approximation to the inverse of the Hessian using

$$B_{k+1}^{-1} = (\mathbf{I} - \rho_k s_k y_k^T) B_k^{-1} (\mathbf{I} - \rho_k y_k s_k^T) + \rho_k s_k s_k^T$$

where $\rho_k = \frac{1}{y_k^T s_k}$. This matrix is certainly positive definite if $\rho_k > 0$ and we've argued in class that that will be the case if the step length is chosen according to the Wolfe conditions. Show that indeed $\rho_k > 0$ for *all* step lengths if $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly convex function. **(2 points)**

If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, powerpoint slides vs whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!