MATH 417: Numerical Analysis

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Homework assignment 7 – due 3/29/2007

Problem 1 (Convergence of Richardson iteration). Richardson iteration is another variant of the methods presented. It can be written in the form

$$x^{k+1} = x^k - \omega\big(Ax^k - b\big),$$

where $\omega > 0$ is called the damping factor. Follow these steps to prove that this method yields a contraction with respect to the norm $\|.\|_2$ if A is a symmetric positive definite matrix and $0 < \omega < 1/\lambda_{\max}$, where $\lambda_{\max}$ is the largest eigenvalue of A.

a) State what the operator $T = \mathbf{1} - BA$ of the iteration is, by identifying what B is for the iteration above.

b) Determine the eigenvalues (and if necessary eigenvectors) of this matrix T.

c) In class we saw that an iterative scheme converges if $\|\mathbf{1} - BA\| \le k < 1$. Use the decomposition of part b) to show this property in the l_2 matrix norm, i.e. to show that $\|\mathbf{1} - BA\|_2 < 1$. For this remember that $\|T\|_2 = \lambda_{\max}(T)$, where $\lambda_{\max}(A)$ denotes the largest eigenvalue of the matrix T.

c) Draw your conclusions concerning the contraction property of T, i.e. show whether or not $\|T\|_2 = \|\mathbf{1} - BA\|_2 < 1$ for the given choice of ω.

(4 points)

Problem 2 (Gauss-Seidel iteration). Repeat Problem 1 of Homework 6 (the Jacobi iteration), but use the Gauss-Seidel iteration instead to compute the vectors $x^{(k)}$. Generate the same plots as before. Compare your results with the previous results. **(4 points)**

Problem 3 (Condition numbers). Calculate the condition numbers $\kappa(A) = \|A\|\,\|A^{-1}\|$ with respect to the l_1, l_∞ and l_2 norms for the matrix

$$A = \begin{pmatrix} 1 & 1.001 \\ 0.999 & 1 \end{pmatrix}.$$

(5 points)

Problem 4 (Error propagation). With the matrix from Problem 3, consider the solutions $x, \tilde{x}$ of the following linear systems:

$$
\begin{aligned}
Ax &= b, & \qquad b &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
A\tilde{x} &= \tilde{b}, & \qquad \tilde{b} &= \begin{pmatrix} 1 \\ 1.001 \end{pmatrix}.
\end{aligned}
$$

(Imagine the former to be the exact right hand side, and the latter to be one that is contaminated by measurement uncertainty, statistical error, etc.)

Solve for x and $\tilde{x}$. Calculate the relative difference in the right hand side $\epsilon_r = \|b - \tilde{b}\|/\|b\|$ and the relative error $e_r = \|x - \tilde{x}\|/\|x\|$ in the solution, each for both the l_2 and the l_∞ norm.

Using your result from Problem 3, do ϵ_r and e_r satisfy the estimates discussed in class, i.e. that $e_r \leq \kappa(A)\epsilon_r$? **(5 points)**

Problem 5 (Matrix norms). Prove that if $\|A\|$ is a matrix norm induced by a vector norm $\|v\|$, then $\|\mathbf{1}\| = 1$ where $\mathbf{1}$ is the identity matrix.

We have shown that induced matrix norms indeed satisfy the three norm conditions. However, there may be other matrix norms that are not induced and that nevertheless also satisfy the norm conditions. Do you think there can be matrix norms that satisfy the norm conditions but for which $\|\mathbf{1}\| \neq 1$? If so, give an example. **(3 points)**