# MATH 412: Theory of Partial Differential Equations


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## Homework assignment 8 – due Thursday 11/1/2007

**Problem 1 (Wave equation).** The wave equation with constant coefficients and zero right hand side reads in one space dimension

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0,$$

where $c$ is the so-called wave speed. Show that if $u$ has the form $u(x,t) = f(x-ct)$ for an arbitrary function $f(s)$, then $u(x,t)$ is a solution of the wave equation. Show that the same is true for $u(x,t) = g(x+ct)$. How about $u(x,t) = \alpha f(x-ct) + \beta g(x+ct)$? **(4 points)**

**Problem 2 (Wave equation).** Solve problem 4.2.1 in the book. **(3 points)**

**Problem 3 (Wave equation).** Solve problem 4.4.1 in the book, parts a) and b). Note that the answers are given in the appendix, but you should explain how you arrive there. **(3 points)**

**Problem 4 (Wave equation).** The total kinetic and potential energy of a vibrating string of length $L$ at any given time $t$ is

$$E(t) = \frac{1}{2}\int_0^L \left(\frac{\partial u(x,t)}{\partial t}\right)^2 + c^2 \left(\frac{\partial u(x,t)}{\partial x}\right)^2 \, dx.$$

Show that the energy is conserved, i.e. that $E(t_1) = E(t_2)$ for any two time instants $t_1, t_2$ if $u(x,t)$ satisfies the homogenous wave equation

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} &= 0, && \text{in } \Omega \times [0,T], \\ u(0,t) &= 0 && \text{for } t \in [0,T], \\ u(L,t) &= 0 && \text{for } t \in [0,T]. \end{aligned}$$

Hint: First note that $E(t_2) - E(t_1) = \int_{t_1}^{t_2} \frac{\partial}{\partial t} E(t) \, dt$. Then derive what form $\frac{\partial}{\partial t} E(t)$ has by direct differentiation under the integral in the definition of $E(t)$.

Then integrate by parts in space and time as necessary and see if you can cancel terms using the wave equation and its boundary values as stated above.

**(4 points)**

**Problem 5 (Wave equation).** Go back in your notes and check on how we showed uniqueness for the heat equation. There, we used the "energy method", where we multiplied the PDE that has to hold for the difference $\delta(x,t) = u_1 - u_2$ of any two solutions, by $\delta$ itself, then integrated over time and space, and finally integrated by parts the spatial component.

Attempt to show uniqueness of solutions of the wave equation in 1d,

$$\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} &= Q(x,t), && \text{in } \Omega \times [0,T], \\
u(0,t) &= 0 && \text{for } t \in [0,T], \\
u(L,t) &= 0 && \text{for } t \in [0,T], \\
u(x,0) &= f(x) && \text{in } \Omega, \\
\frac{\partial}{\partial t} u(x,0) &= g(x) && \text{in } \Omega.
\end{aligned}$$

Follow the same steps as before: First, derive the equation that $\delta(x,t)$ has to satisfy. In contrast to the heat equation, next multiply by $\frac{\partial}{\partial t}\delta(x,t)$ instead of by $\delta$, integrate over space and time, and integrate by parts with respect to space. See if you can derive uniqueness of solutions from this. **(5 points)**