Projects

Renzo’s math 472

The list of questions/projects below will lead the discussion for this week. Don’t be shy, put forth your ideas - Josh is here to help you!

1 Dense Sets

Let $X$ be a topological space. A set $A$ is called dense if its closure is $X$.

**Problem 1.** Give some example of some dense sets in some familiar topologies. What sets are dense in the discrete topology?

   Consider the finite complement topology on a set $X$, where the open sets are $\emptyset$, $X$ and those sets such that their complement is finite.

**Problem 2.** What sets are dense in the finite complement topology? Make sure to discuss both the case where $X$ is a finite or an infinite set.

**Problem 3** (The Zariski Topology). Consider the following topology on the plane: a set is defined to be closed if it the zero set of a system of polynomials in two variables, i.e. $C = \{(x, y) \text{ such that } f_1(x, y) = f_2(x, y) = \cdots = 0\}$.

1. Verify that this is indeed a topology on the plane.

2. Check that all open sets that are not $\emptyset$ are dense.

3. Define in the appropriate way the Zariski topology on the line. What topology is it? (i.e. have you seen it before?)

2 Base for a Topology

Consider a topological space $(X, \tau)$. Recall that a base for a topology is a subset $\beta \subseteq \tau$ such that any open set in $\tau$ can be obtained as the union of sets in $\beta$.

**Problem 1.** Show that alternatively, one can define a base of a topology by requiring that for any point $x \in X$ and any open set $O \ni x$, there exists a set $B \in \beta$ that contains $x$ and is contained in $O$.

**Problem 2.** Show that open balls of any radius form a basis for the euclidean topology on $\mathbb{R}^n$. How about open cubes?
Problem 3. Show that if $X$ has a countable base for its topology, then $X$ contains a countable dense subset.

3 Separating Stuff

A topological space $X$ is called Hausdorff if, for any pair of points $x, y \in X$, there are two open sets $O_x, O_y$ such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$.

Problem 1. Show that if a space $X$ has a metrizable topology, then it is Hausdorff.

A topological space is called T1 if for any pair of points $x, y \in X$, there is an open set $O_x$ such that $x \in O_x$, $y \not\in O_x$.

Problem 2. Show that being T1 is equivalent to the fact that points are closed sets.

Problem 3. Can you think of a topological space that is T1 but not Hausdorff?

4 Product spaces

Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces. We define a new topological space $X \times Y$, called the product (space) of $X$ and $Y$. We define a natural topology on $X \times Y$, starting from $\tau$ and $\sigma$, called the product topology.

$X \times Y$ as a set: the points of $X \times Y$ correspond to ordered pairs $(x, y)$, where $x \in X$ and $y \in Y$.

A basis for the product topology: a basic open set $B \in \beta$ is a set of the form $U \times V$ (i.e. the set of ordered pairs $(u, v)$, with $u \in U$ and $v \in V$). $\beta$ is a base for a topology called the product topology.

Problem 1. Familiarize yourself with the product topology. Draw some pictures, make some examples, understand what the basic open sets look like, and find some examples of open sets in the product topology that are not basic opens.

Recall that you have two natural maps from a product space called the projections:

$$
\pi_X : X \times Y \longrightarrow X \\
(x, y) \mapsto x
$$

$$
\pi_Y : X \times Y \longrightarrow Y \\
(x, y) \mapsto y
$$
**Problem 2.** Prove that the product topology is the coarsest topology that makes both projection maps continuous.

**Problem 3.** Show that the torus and the cylinder are product spaces. Show that the product topology coincides with the induced topology from the euclidean topology in $\mathbb{R}^3$.

Prove the following theorems:

**Theorem 1.** $X \times Y$ is Hausdorff if and only if both $X$ and $Y$ are Hausdorff.