Solutions to homework 1

**Question 1:** prove that a function \( f : X \rightarrow Y \) is continuous (calculus style) if and only if the preimage of any open set in \( Y \) is open in \( X \).

**Proof:**

First, assume that \( f \) is a continuous function, as in calculus; let \( U \) be an open set in \( Y \), we want to prove that \( f^{-1}(U) \) is open in \( X \).

If \( p \) is a point in \( f^{-1}(U) \), we must show there is a little open ball around \( p \) that is all contained in \( f^{-1}(U) \).

But \( f(p) \in U \) which is an open set, so there exists a ball \( B \) of radius \( r \) centered at \( f(p) \) and all contained in \( U \).

Continuity calculus style tells us that provided that we take a small enough radius, there is a ball \( C \) around \( p \) such that \( f(C) \) is contained in \( B \), and hence in \( U \). Which means that \( C \) is all contained in \( f^{-1}(U) \). So we are done with one side of the proof.
Now assume that for any open set in $Y$, its preimage via $f$ is open. We want to show that $f$ is a continuous function. Let $p$ be a point in $X$, $f(p)$ the corresponding image in $Y$.

To show that $f$ is continuous at $p$ we must show that, given a ball $B$ of radius $\varepsilon$ around $f(p)$, there exists a ball $C$ whose image is entirely contained in $B$.

But $B$ in particular is an open set. Therefore $f^{-1}(B)$ is open. Therefore $p$ is an interior point for $f^{-1}(B)$: there is a little ball $C$ centered at $p$ contained in $f^{-1}(B)$.

This implies that $f(C)$ is contained in $B$, which is what we needed to show.

**Question 2:** prove that a function $f : X \rightarrow Y$ is continuous (calculus style) if and only if the preimage of any closed set in $Y$ is closed in $X$.

**Proof:** We want to exploit the previous exercise, and the fact that the complement of an open set is closed.
Assume $f$ is continuous.

Let $K$ be any closed set in $Y$.

Then $Y \setminus K$ is open.

Then $f^{-1}(Y \setminus K)$ is open by exercise 1.

But $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$.

Hence $f^{-1}(K)$ is closed.

Now assume the preimage of any closed set is closed.

Let $U$ be any open set in $Y$.

$Y \setminus U$ is closed.

Hence $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed.

Which implies that $f^{-1}(U)$ is open. Hence the preimage of any open set is open, and $f$ is continuous by exercise 1.