Rien n’est beau que le vrai. - Hermann Minkowski

1. Read S30, S31.

2. S30.4, S31.1


4. Find the continued fraction expansion of \((1+\sqrt{5})/2\). What is the relationship between the convergents of this continued fraction and the Fibonacci numbers.

5. Demonstrate that the continued fraction of \(\sqrt{11}\) is \(\{3, 3, 6\}\). Find the convergent with smallest denominator which approximates \(\sqrt{11}\) to within \(1/1000\).

6. Find the value of each of the following infinite purely periodic continued fractions: \(\{1, 2, 3\}\), \(\{3, 2, 1\}\), \(\{1, 2\}\), \(\{2, 1\}\), \(\{2, 3, 1\}\), \(\{1, 3, 2\}\). Any conjectures?

7. Prove the remark on page 113 Davenport: The recurrence relations (8) and (9) are equivalent to the fact that the vector from \(P_{n-2}\) to \(P_n\) is an integral multiple of the vector from the origin \(O\) to \(P_{n-1}\).

8. Show that if \(p > q\) and the continued fraction for \(p/q\) is \(\{q_0, q_1, \ldots, q_n\}\) then the continued fraction for \(q/p\) is \(\{0, q_0, q_1, \ldots, q_n\}\).

9. Suppose \(q_0 \geq 1\). Determine all rational numbers \(p/q\) such that the first two convergents of \(p/q\) are \(C_0 = q_0\) and \(C_1 = q_0 + 1\).

10. Find the first two non-integer best approximations for the transcendental number \(e\).

11. Show that if \(x^3 - 2y^3 = 1\) for some \(x, y \in \mathbb{Z}\), then \(x/y\) is a convergent of \(2^{1/3}\). Hint: bound \(|x/y - 2^{1/3}|\) and use class lemma. Find a non-trivial solution to this equation.

12. Suppose that \(p \equiv 1 \mod 4\). Show that \(x^2 - py^2 = -1\) has a solution for some \(x, y \in \mathbb{Z}\):
   a) Hints: Let \(a, b \in \mathbb{Z}\) be the smallest solution to \(a^2 - pb^2 = 1\). Show that \(a = 2a_1 + 1\) and \(b = 2b_1\) for some \(a_1, b_1 \in \mathbb{Z}\).
   b) Show that \(a_1(a_1 + 1) = pb_1^2\). Note that \(gcd(a_1, a_1 + 1) = 1\). Explain why one of \(a_1\) and \(a_1 + 1\) must be \(p\) times a perfect square and the other must be a perfect square. Conclude that \(a \pm 1 = 2u^2\) and \(a \mp 1 = 2pv^2\) for some \(u, v \in \mathbb{Z}\).
   c) Show that \(u^2 - pv^2 = \pm 1\) and explain why \(u^2 - pv^2 = +1\) is not possible.
   d) Use table on page 206 Silverman to find a solution to \(x^2 - 53y^2 = -1\).