Pries: Combinatorial Number Theory

Homework 7: Monday, April 21, 2002

Mathematical genius and artistic genius touch one another - Gosta Mittag-Leffler

1. Read Silverman pages 341-364, Silverman-Tate 38-41, 47-48, 145-149.

2. F41.2 (a,c), F41.4 (hint: (a) look mod 4), F42.2, F42.5, F43.1, F43.4 (a,b)

3. Suppose $P = (x, y)$ is a point on $y^2 = x^3 + ax^2 + bx + c$.
   a) Verify that the $x$-coordinate of $2P$ is given by the formula on pg 39 of S-T.
   b) Find a formula for the $y$-coordinate of $2P$.
   c) Verify statement c) on page 40 of Silverman-Tate.
   d) Find all points of order 3 on the elliptic curve $y^2 = x^3 + 1$ using complex numbers.

4. If $t \neq 1, t \neq 1/4, t \in \mathbb{Q}$, show that $P = (t, t)$ is a rational point of order 4 on $E : y^2 = x^3 - (2t - 1)x^2 + t^2x$. Hint: look at $2P$.

5. Consider the point $P = (3, 8)$ on $E : y^2 = x^3 - 43x + 166$.
   a) Compute $2P$, $4P$ and $8P$. What conclusion can you draw comparing $P$ with $8P$?
   b) Show that 3 does not divide $\Delta(E)$.
   c) Find the number of points of $E(\mathbb{F}_3)$; don’t forget the point at infinity!
   d) Use the Reduction Mod $p$ Theorem to find the number of points on $E(\mathbb{Q})$.

6. Recall that $E : f(x, y) = y^2 - f(x)$ is non-singular if it has no points $P = (x, y)$ such that $f(x, y) = f_x(x, y) = f_y(x, y) = 0$.
   a) Prove the two formulas for $D = \Delta(E)$ on pg 47 Silverman-Tate are the same.
   b) Prove that $E$ is non-singular if and only if $\Delta(E) = 0$. Hint: Remember what we proved in class about how to characterize polynomials $f(x)$ with double roots.

7. Find all rational points of finite order on $y^2 = x^3 - 2$.

8. Let $p$ be a prime. Find all rational points of finite order on $y^2 = x^3 + px$.

9. Suppose $(x, y)$ is a $\mathbb{Z}$-rational point on $x^3 + xy^2 = m$. If $m = AB$, find conditions on $A$ and $B$ under which there is an integer solution $(x, y)$. Prove that $\max\{x^2, y^2\} \leq 1 + m^2$. Find all integer solutions to $x^2y + y^2x = 240$. 