Mathematics V1205y, Calculus IIIS/IVA

Sample Final:

There is a reasonably possibility of errors on this document, since I needed to finish it very quickly.

Name: ____________________________

1. Evaluate \( \int_0^1 \int_{\sqrt{y}}^{1} \sqrt{x^3 + 1} \, dx \, dy \).
   
   \( 2(2^{3/2} - 1)/2 \)

2. Find the center of mass of the lamina that occupies the quarter-circle \( D = \{ x^2 + y^2 \leq 1, x \geq 0, y \geq 0 \} \) if the density is proportional to the square of the distance to the origin.
   
   \( (8/5\pi, 8/5\pi) \).

3. Icecream fills the space below the sphere \( x^2 + y^2 + z^2 = a^2 \) and above the cone \( z = \sqrt{x^2 + y^2} \). The density of the icecream is given by the function \( f(x, y, z) = z \).
   Find the mass of the icecream using spherical coordinates.
   
   \( \pi a^4 / 8 \).

4. Evaluate the integral \( \int \int_{R} \sin(9x^2 + 4y^2) \, dA \) where \( R \) is the region in the \( xy \)-plane bounded by \( 9x^2 + 4y^2 = 1 \).
   
   \( \pi(1 - \cos(1))/24 \).

5. Let \( F(x, y) = -\frac{y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j} \) and let \( C \) be the curve \( x^2 + y^2 = 1 \) oriented clockwise. Evaluate the line integral \( \int_C F \cdot dr \).
   
   \( 2\pi \)

6. Find the simple closed curve \( C \) which gives the maximal value of the following integral and explain why it yields the maximal value:
   
   \( \int_C (x^5 - 6y + y^3) \, dx + (y^4 + 6x - x^3) \, dy \).
   
   Green: \( x^2 + y^2 \leq 4 \).

7. Consider the vector field \( F(x, y, z) = 2xi + 2yj + 2zk \).
   a) Compute \( \text{curl}(F) \).
   
   b) If \( C \) is any path from \((0, 0, 0)\) to \((a_1, a_2, a_3)\) and \( a = a_1i + a_2j + a_3k \), prove that \( \int_C F \cdot dr = a \cdot a \).
   
   0, Down 1-to-0 thm.
8. Sketch the surface given parametrically by \( r(u, v) = (\cos(v), \sin(v), u) \) over the domain \( D = \{-1 \leq u \leq 1, 0 \leq v \leq 2\pi\} \). Find the normal vector at the point when \( (u, v) = (0, \pi/4) \).

pg 1091 IV, \( n = (\sqrt{2}/2, \sqrt{2}/2, 0) \).

9. Suppose \( F(x, y, z) = zk \) and \( S \) is the part of the plane \( z = x \) lying over the square \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \). Find the flux of \( F \) across \( S \).

\( 1/2 \).

10. Suppose \( F(x, y, z) = (x, y, z) \) and \( S \) is the surface \( x^2 + y^2 + z^2 = 2 \) with \( z \geq 0 \). Find \( \iint_S F \cdot dS \).

\( 4\sqrt{2}\pi \)

11. Suppose \( F(x, y, z) = (xe^z - 3y)i + (ye^{z^2} + 2x)j + (x^2y^2z^2)k \) and \( S \) is the portion of the paraboloid \( z = 4 - x^2 - y^2 \) where \( z \geq 0 \). Compute \( \iint_S \text{curl}(F) \cdot dS \).

\( 20\pi \)

12. Let \( E \) be the region enclosed by the paraboloid \( z = 2 - x^2 - y^2 \) and the plane \( z = 1 \). Let \( S \) be the surface bounding \( E \). Let \( F(x, y, z) = \langle z \tan^{-1}(y^2), z^3\ln(x^2 + 1), z \rangle \). Find the flux of \( F \) across \( S \); in other words find \( \iint_S F \cdot dS \).

\( 3\pi/2 \).

13. If \( a + bi = (\sqrt{3} + i)^11 \), solve for \( a \) and \( b \).

\( a = 2^{10}\sqrt{3}, b = -2^{10} \).

14. Suppose \( f(t) = 1/(t^2 - i) \).

A. If \( f(t) = a(t) + bi(t)i \), solve for \( a(t) \) and \( b(t) \).

B. Find the absolute value \( |f(t)| \). For which value of \( t \) is \( |f(t)| \) maximal?

C. Sketch a parametric graph of \( f(t) \) on the complex plane.

\( a(t) = t^2/(t^4 + 1), b(t) = 1/(t^4 + 1) \).

15. Show that the set of functions \( \{f(t) = e^{ikt}\} \) are orthogonal.

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16. Given two complex vectors \( \vec{v}_1 \) and \( \vec{v}_2 \), prove that \( w_1 = v_1 \) and \( w_2 = v_2 - \text{proj}_{w_1}v_2 \) are orthogonal.

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17. Find the best line approximating \( f(x) = x^3 \) on the interval \( [1, 2] \). Hint: using the orthonormal basis \( w_1 = 1 \) and \( w_2 = \sqrt{12}(x - 3/2) \) with respect to the inner product \( \langle f(x), g(x) \rangle = \int_1^2 f(x)g(x)dx \), it is only necessary to compute two integrals.

Simplify \( \left( \int_1^2 x^3dx \right)1 + \left( \int_1^2 x^3 \sqrt{12}(x - 3/2)dx \right)\sqrt{12}(x - 3/2) \).
18. A. Find the complex fourier series of $f(t) = \sin^3(t)$.

B. Find the real fourier series of $f(t) = \sin^3(t)$.

A. $(-1/8i)e^{3it} + (3/8i)e^{it} + (-3/8i)e^{-it} + (1/8i)e^{-3it}$.

B. $(-1/4)\sin(3t) + (3/4)\sin(t)$.

19. Suppose $f(t)$ is the function of period $2\pi$ such that $f(t) = t/2$ if $-\pi < t < \pi$. Find the real fourier series of $f(t)$ from scratch.

$$RFS = \sum_{n=1}^{\infty} [(-1)^{n+1}/n] \sin(nt).$$

20. Suppose $f(t)$ is the function of period $2\pi$ such that $f(t) = 0$ if $-\pi < t < 0$ and $f(t) = t$ if $0 < t < \pi$. To what value does the real fourier series of $f(t)$ converge when $t = \pi$. Hint: this is an extremely easy short problem.

$\pi/2$. 