The Number $e$

Geometrically, the number $e$ is defined to be that positive value of $a$ for which the graph of the exponential function $y = f(x) = a^x$ has slope = 1 at $x = 0$.

The slope being the derivative, the slope condition on $a$ is equivalent to this limit condition:

$$e \text{ is that value of } a \text{ for which } \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

The number $e$ satisfies another limit condition, which is essential in the theory of compound interest:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

The proof of this second limit goes as follows.

Let $n$ be a positive integer. Then we first show that the terms of the sequence $x_n = (1 + \frac{1}{n})^n$ approach a finite limit as the values $n = 1, 2, 3, \ldots$ increase towards $\infty$. And then we show that this limit equals $e$.

(1) For the first step, we only give the outline of the argument, because it involves quite a bit of algebra. We first expand the expression $(1 + \frac{1}{n})^n$ by means of the binomial theorem into a sum of $n + 1$ terms.

Then by looking at the terms of this sum, and by cleverly rewriting the expression for the general $k$-th term of the sum, it can be seen that each succeeding $x_n$ is strictly bigger than the previous one, and also that all $x_n$ are bigger than 2. By comparing the general $k$-th term to the terms $\frac{1}{2^k}$, and by considering sums of such terms, it can be seen that all $x_n$ are smaller than 3.

So the terms of the sequence $x_n$ have to “pile up” at a limiting value $x_*$, which has to lie between the numbers 2 and 3.

(2) For the second step, we want to show that this number $x_*$ coincides with the number $e$ defined by the above slope condition on the exponential function.

The argument goes as follows. First, set $t = a^h - 1$. Then $a^h = 1 + t$, and hence $h = \log_a a^h = \log_a (1 + t)$. Then we get

$$\frac{a^h - 1}{h} = \frac{t}{\log_a (1 + t)} = \frac{1}{t} \log_a (1 + t) = \log_a (1 + t)^{1/t} = \frac{1}{\log_a (1 + u)^u},$$

where $u = \frac{1}{t}$. Now let $u \to \infty$; then the right side of the above sequence of equations goes to $\frac{1}{\log_a x_*}$. Since $u \to \infty$ implies that $t \to 0$ and hence $h \to 0$, we see that if we put $a = e$, then the left side goes to 1.

We then conclude that $\log_e x_* = 1$, which says that $x_* = e^1 = e$. That concludes the argument.