Metric Spaces

A metric space is a collection, \( X \), of objects \( x, y, z, \ldots \) together with a distance function, \( d(x, y) \), that associates to any two objects \( x, y \), in the space, a real number called the "distance between \( x \) and \( y \)". The distance function must have the following properties:

**Definition** - The function \( d : X \times X \to \mathbb{R} \) is a **distance function** if

1. \( a. \) \( d(x, y) = d(y, x) \) \( \forall x, y \in X \)
2. \( b. \) \( d(x, y) \geq 0 \) and \( d(x, y) = 0 \) if and only if \( x = y \) \( \forall x, y \in X \)
3. \( c. \) \( d(x, y) \leq d(x, z) + d(z, y) \) \( \forall x, y, z \in X \)

We consider some examples of metric spaces:

1. Let \( X = (-\infty, \infty) \), the real numbers with \( d(x, y) = |x - y| \). Clearly this \( d(x, y) \) satisfies all the necessary conditions for a distance function and it follows that the real numbers is one example of a metric space.

2. More generally, \( X = \mathbb{R}^n \), the space of \( n \)-tuples of real numbers with
   \[
   d(x, y) = \left[ \sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2}
   \]
   is another example of a metric space. Part c of the definition has to be checked in this case.

Write first

\[
\begin{align*}
   d(x, y) & = \left[ \sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2} \\
            & \leq d(x, z) + d(z, y) \\
            & = \left[ \sum_{i=1}^{n} (x_i - z_i)^2 \right]^{1/2} + \left[ \sum_{i=1}^{n} (z_i - y_i)^2 \right]^{1/2}
\end{align*}
\]

Then let \( a_i = x_i - z_i \) and \( b_i = z_i - y_i \) so that \( x_i - y_i = a_i + b_i \)

and write

\[
\sum_{i=1}^{n} (x_i - y_i)^2 = \sum_{i=1}^{n} (a_i + b_i)^2
\]

\[
= \sum_{i=1}^{n} a_i^2 + 2 \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2
\]

\[
\leq \sum_{i=1}^{n} a_i^2 + 2 \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2} + \sum_{i=1}^{n} b_i^2
\]
Here we used the Cauchy-Schwartz inequality for n-tuples,
\[ \sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2} \]

Then
\[ \sum_{i=1}^{n} (x_i - y_i)^2 \leq \sum_{i=1}^{n} a_i^2 + 2 \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2} + \sum_{i=1}^{n} b_i^2 \]
\[ \leq \left[ \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} + \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2} \right]^2 \]

which is
\[ d(x, y)^2 \leq [d(x, z) + d(z, y)]^2. \]

Then \( X = \mathbb{R}^n \), the space of n-tuples of real numbers with \( d(x, y) = \left[ \sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2} \) satisfies all the conditions of a metric space.

3. Let \( X = C[a, b] \), the space of functions that are defined and continuous on \( I = [a, b] \). We can define \( d(x, y) = \max_{x \in I} |x(t) - y(t)| \). It is not difficult to verify that this definition of \( d(x, y) \) satisfies all the conditions for a distance function and it follows that this is a metric space.

4. Let \( X = C[a, b] \), the space of functions that are defined and continuous on \([a, b]\). We can define \( d(x, y) = \left( \int_a^b |x(t) - y(t)|^2 \, dt \right)^{1/2} \). It is not difficult to verify that this definition of \( d(x, y) \) satisfies the conditions \( a \) and \( b \) for a distance function and, using the Cauchy-Schwartz inequality for functions on \([a, b]\) that condition \( c \) holds as well, Then it follows that this is a metric space.

**Problem** Mimic the proof that \( d(x, y) = \left[ \sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2} \) satisfies condition \( c \) to show that 
\[ d(x, y) = \left( \int_a^b |x(t) - y(t)|^2 \, dt \right)^{1/2} \]

satisfies this condition. The analogue of the C-S inequality for n-tuples is
\[ \left| \int_a^b x(t) y(t) \, dt \right| \leq \left( \int_a^b x(t)^2 \, dt \right)^{1/2} \left( \int_a^b y(t)^2 \, dt \right)^{1/2} \]

**Definition** A metric space \( X \) is said to be complete if every Cauchy sequence of elements in \( X \) converges to an element of \( X \)

Not every metric space need be complete as the following examples illustrate:

1. Let \( X = \mathbb{R} \), the real numbers with \( d(x, y) = |x - y| \). Since every Cauchy sequence of real numbers \( \{x_n\} \) converges to a limit in \( \mathbb{R} \), this is a complete metric space. On the other hand, let \( X = \mathbb{Q} \), the rational numbers with the same distance function. Now, there are Cauchy sequences of rationals that converge to an irrational limit. Therefore this metric space is not complete.

2. Let \( X = C[a, b] \), the space of functions that are defined and continuous on \( I = [a, b] \). We can define \( d(x, y) = \max_{x \in I} |x(t) - y(t)| \). Then \( \{f_n(x)\} \subset X \) is a Cauchy sequence if \( d(f_n, f_m) \to 0 \)
Let \( m,n \to \infty \), i.e., for every \( \varepsilon > 0 \) there exists \( N > 0 \) such that
\[
d(f_n,f_m) = \max_{x \in I} |f_n(x) - f_m(x)| < \varepsilon \quad \text{for all} \quad m,n > N.
\]
Suppose then that \( \{f_n(x)\} \subset X \) is a Cauchy sequence in \( X \). Then for each fixed \( x \) in \( I \), \( \{f_n(x)\} \) is a Cauchy sequence of real numbers and thus this sequence converges to a limit which we will denote by \( f(x) \). Now define \( f(x) = \lim_{n \to \infty} f_n(x) \) for each \( x \) in \( I \). From this definition of \( f \) it is evident that \( d(f_n,f) \to 0 \) as \( n \to \infty \). It remains to show that \( f \in X \).

For arbitrary \( x,y \) in \( I \) we have
\[
|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\
\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
\leq |f(x) - f(y)|.0
\]

Let \( \varepsilon > 0 \) be given. Since \( d(f_n,f) = \max_{x \in I} |f_n(y) - f(y)| \to 0 \) as \( n \to \infty \) it follows that there exists an \( N > 0 \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( n > N \), and this holds for all \( x \) and \( y \) in \( I \); i.e., \( N \) is independent of \( x,y \) in \( I \). Now with an \( n > N \) fixed,
\[
|f(x) - f(y)| \leq 2\varepsilon + |f_n(x) - f_n(y)|,
\]
and since each \( f_n \) in \( X \) is uniformly continuous, there exists a \( \delta > 0 \) such that \( |f_n(x) - f_n(y)| < \varepsilon \) if \( |x - y| < \delta \). We conclude that for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |f(x) - f(y)| < 3\varepsilon \) and it follows that \( f \in X \).

3 Let \( X = C[-1,1] \), the space of functions that are defined and continuous on \([-1,1] \), with the distance function \( d(x,y) = \left( \int_{-1}^{1} |x(t) - y(t)|^2 dt \right)^{1/2} \). We will show that with this distance function, \( X \) is not complete, that there are Cauchy sequences that converge to a limit that does not belong to \( X \). One such sequence is the sequence \( \{f_n(x)\} \) given by
\[
f_n(x) = \begin{cases} 
-1 & \text{if} \quad -1 \leq x \leq -1/n \\
nx & \text{if} \quad -1/n \leq x \leq 1/n \\
+1 & \text{if} \quad 1/n \leq x \leq 1 
\end{cases}
\]

The following sketch shows \( f_2 \) and \( f_6 \)

![Sketch of functions](image)

\[ f_2(x) \text{ and } f_6(x) \text{ vs } x \]

In general, it is not hard to show that for \( n > m > 0 \),
\[ |f_n(x) - f_m(x)| = \left( \frac{1}{n} - \frac{m}{n^2} \right) + \left( 1 - \frac{m}{n} \right) \left( \frac{1}{m} - \frac{1}{n} \right) = \frac{1}{n} \left( 1 - \frac{m}{n} \right) + \frac{1}{m} \left( 1 - \frac{m}{n} \right)^2 \]

and this leads to \( \int_{-1}^{1} |f_n(x) - f_m(x)|^2 \to 0 \) as \( m, n \to \infty \). Then this is a Cauchy sequence in the sense of this distance function but the limit of this sequence is the function \( f(x) = \text{sgn}(x) \) which is not continuous at \( x = 0 \). Then this metric space is not complete.

Another example of a Cauchy sequence in \( X = C[-1, 1] \) which converges to \( \text{sgn}(x) \) is the sequence \( \{g_n(x)\} = \{\arctan nx\} \). It can be seen from the following sketch that these smooth functions appear to converge to the same limit as the functions in the previous example.

![Sketch of functions](image)

Even when a metric space is not complete, it is always possible to include it in a larger, complete metric space. For example, the rationals with \( d(x, y) = |x - y| \) is not complete but is contained in the space of real numbers with the same metric and this larger space is complete as we know.

**Definition** A metric space \( X^* \) is said to be the completion of the metric space \( X \) if \( X \) is contained in \( X^* \) and if every Cauchy sequence in \( X \) converges to an element of \( X^* \).

**Theorem** Every metric space has a completion and all of its completions are isometric

The term "isometric" means that any two completions of the same \( X \) differ at most in the names that the elements are given. We will not give the proof of this theorem.

The space of functions \( X = C[-1, 1] \), defined and continuous on \([-1, 1]\), with the distance function \( d(x, y) = \left( \int_{-1}^{1} |x(t) - y(t)|^2 \, dt \right)^{1/2} \) has been shown to be not complete. The completion of this space is the space we call the space of "square integrable functions", denoted by \( L^2[-1, 1] \). This space of fundamental importance in more advanced courses in analysis.
The following theorem is often useful in applications.

**Theorem** (Fixed Point Theorem) Suppose $f$ is a function defined on a complete metric space $X$ with values in $X$ such that $d(f(x), f(y)) \leq k d(x, y)$ for all $x, y$ in $X$ with $0 < k < 1$. Then the function $f$ has a unique fixed point $x_0$ in $X$; i.e. $f(x_0) = x_0$. In addition, if $x_1$ is any element of $X$ define a sequence $\{x_n\}$ by $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$. Then $\{x_n\}$ converges to $x_0$.

**Proof** - Start with any $x_1$ in $X$ and set $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$. Then

$$d(x_2, x_3) = d(f(x_1), f(x_2)) \leq k d(x_1, x_2)$$
$$d(x_2, x_3) = d(f(x_1), f(x_2)) \leq k d(x_1, x_2)$$
$$d(x_3, x_4) = d(f(x_2), f(x_3)) \leq k d(x_2, x_3) \leq k^2 d(x_1, x_2)$$

In general,

$$d(x_n, x_{n+1}) \leq k^{n+1} d(x_1, x_2) \quad \text{for} \quad n = 1, 2, \ldots \quad (\ast)$$

Now let $m, n$ be any positive integers with $m > n$. Then by the triangle inequality

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m).$$

and it follows from repeated application of $(\ast)$ that

$$d(x_n, x_m) \leq d(x_1, x_2)[k^{n-1} + k^n + k^{n+1} + \ldots + k^{m-2}]$$
$$d(x_1, x_2)[k^{n-1} + k^n + k^{n+1} + \ldots + k^{m-2}]$$

Since $\sum_{j=0}^{\infty} k^j = \frac{1}{1-k}$ we find

$$d(x_n, x_m) \leq d(x_1, x_2) k^{n-1} \frac{1}{1-k}$$

Let $\varepsilon > 0$ be given. Since $0 < k < 1$, the right side of this last expression tends to zero as $n \to \infty$. Then there is an $N$ such that the right side is less than $\varepsilon$ for all $n > N$. But this means \{x_n\} is a Cauchy sequence and since $X$ is complete, there exists $x_0$ in $X$ such that $x_n \to x_0$. Since $d(f(x), f(y)) \leq k d(x, y)$, it follows that $f$ is continuous on $X$ and so $f(x_n) \to f(x_0)$ as $n \to \infty$. But $f(x_n) = x_{n+1} \to x_0$ and it follows that $f(x_0) = x_0$.

To see that $x_0$ is unique, suppose there exists $y_0$ in $X$ such that $f(y_0) = y_0$. In that case $d(x_0, y_0) = d(f(x_0), f(y_0)) \leq k d(x_0, y_0)$. But $0 < k < 1$, which implies that $d(x_0, y_0) = 0$ so $x_0 = y_0$. 

The function $f$ in this theorem is an example of what is called a **contraction mapping** on a complete metric space. Various kinds of problems can be formulated in such a way as to take advantage of the theorem we have just proved. For example consider the problem of finding a solution $y(t)$ for the following initial value problem.
\[
\frac{dy}{dt} = F(t,y(t)) \quad \text{y}(t_0) = y_0
\]

Here we suppose \(F(t,y)\) is continuous in \(t\) for \(-h < t < t_0 + h\), and \(F\) is Lipschitz continuous in \(y\); in some neighborhood of \(y_0\) i.e., for \(y_0 - k < y < y_0 + k\). Let \(R\) denote the rectangle \(\{(t,y) : |t - t_0| < h, |y - y_0| < k\}\). Then

\[
|F(t,y_1) - F(t,y_2)| \leq M_1 |y_1 - y_2| \quad \text{for} \quad (t,y_1), (t,y_2) \in R
\]

Recall that \(F(t,y)\) is Lipschitz in \(y\) if \(F\) has a derivative with respect to \(y\) that is bounded on the rectangle \(R\).

If \(y = y(t)\) solves the initial value problem, then integrating the equation with respect to \(t\) shows that \(y(t)\) also solves the following integral equation

\[
\int_{t_0}^{t} y'(s) ds = y(t) - y_0 = \int_{t_0}^{t} F(s,y(s)) \, ds
\]

i.e. \(y(t) = y_0 + \int_{t_0}^{t} F(s,y(s)) \, ds\)

Let \(I = [t_0 - h, t_0 + h]\) and \(X = C[I]\) with \(d(x,y) = \max|\{x(t) - y(t)\}|\). Then \(X\) is a complete metric space. Since \(F(t,y)\) is assumed to be uniformly continuous throughout some neighborhood of \((t_0,y_0)\), we know that for some constant \(M_0 > 0\), we have \(|F(t,y)| \leq M_0\) on this neighborhood. Then for \(|t - t_0| \leq h\), we have \(|y(t) - y_0| = \left|\int_{t_0}^{t} F(s,y(s)) \, ds\right| \leq M_0 h\) and we will focus on the neighborhood \(\{(t,y) : |t - t_0| \leq h, |y(t) - y_0| \leq M_0 h\}\).

Now we define a function \(G\) on \(X\) with values in \(X\)

\[
G[y(t)] = y_0 + \int_{t_0}^{t} F(s,y(s)) \, ds \quad \text{for} \quad y \in X
\]

If \(M_0 h < k\) and \(M_1 h < 1\) then we can show there exists a function \(y(t)\), continuously differentiable on \(I\), with \(|y(t) - y_0| \leq k\) for \(t \in I\), which satisfies the initial value problem. In particular, this solution is going to be a fixed point of the function \(G\). Note first that if \(|y(t) - y_0| \leq k\) for \(t \in I\) and \(z(t) = G[y(t)]\), then

\[
|z(t) - y_0| = \left|\int_{t_0}^{t} F(s,y(s)) \, ds\right| \leq M_0 h \leq k
\]

i.e., if \((t,y) \in R\) and \(z(t) = G[y(t)]\), then \((t,z) \in R\).

Note also that if \(|y_i(t) - y_0| \leq k\) for \(t \in I\) and \(z_i(t) = G[y_i(t)]\) \(i = 1,2\) then
\[
|z_1(t) - z_2(t)| = \left| \int_{t_0}^{t} [F(s,y_1(s)) - F(s,y_2(s))] \, ds \right|
\leq h \max_{s \in [t_0, t]} |F(s,y_1(s)) - F(s,y_2(s))| \\
\leq hM_1 \max_{s \in [t_0, t]} |y_{n+1}(t) - y_n(t)| \\
\leq M_1 h \, d(y_1, y_2)
\]

Since \( M_1 h < 1 \), it follows that \( d(G(y_1), G(y_2)) = d(z_1, z_2) < d(y_1, y_2) \), which means that \( G \) is a contraction mapping of \( X \) into \( X \). Then the theorem implies that \( G \) has a unique fixed point \( y(t) \in X \), which satisfies \( G[y(t)] = y(t) = y_0 + \int_{t_0}^{t} F(s,y(s)) \, ds \); i.e., the fixed point is a solution of the integral equation.

Note that if \( y(t) \) solves the integral equation then the fundamental theorem of calculus implies that \( y(t) \) is differentiable and

\[
\frac{d}{dt} \left[ y_0 + \int_{t_0}^{t} F(s,y(s)) \, ds \right] = \frac{d}{dt} y(t) \\
F(t,y(t)) = \frac{d}{dt} y(t)
\]

i.e., the solution of the integral equation also solves the original differential equation and \( y(t_0) = y_0 \) so \( y(t) \) solves the initial value problem.