Exam 2  Continuity

1. Use the \( \varepsilon - \delta \) definition of continuity to show that that \( f(x) = x^2 + 1 \) is continuous at \( x = 1 \).

\[
|f(x) - f(1)| = |x^2 + 1 - 2| = |x^2 - 1| = |x + 1||x - 1| \leq \frac{5}{2} |x - 1| \quad \text{if} \quad |x - 1| < \frac{1}{2}
\]

Then \( \forall \varepsilon > 0, \exists \delta = 2\varepsilon/5 \) such that \( |f(x) - f(1)| < \varepsilon \) when \( |x - 1| < \min\left[\frac{1}{2}, 2\varepsilon/5\right] \)

2. Use whatever method you like to show \( f \) is not continuous for all \( x \) if

\[
f(x) = \begin{cases} 
  1 & \text{if } |x - 1| < 1 \\
  0 & \text{if } |x - 1| \geq 1 
\end{cases}
\]

Since \( f(x) = 1 \) if \( 0 < x < 2 \) and \( f(x) = 0 \) otherwise, the sequences \( x_n = 2 - \frac{1}{n} \) and \( y_n = 2 + \frac{1}{n} \) both converge to \( x = 2 \), but \( f(x_n) \to 1 \), while \( f(y_n) \to 0 \). Then by the sequence definition of continuity, \( f \) is discontinuous at \( x = 2 \). A similar argument shows \( f \) is discontinuous at \( x = 0 \).

3. Tell whether the following statements are true or false

If the statement is false, give an example that shows it is false.
If the statement is true, state a result that supports that conclusion.

a) If \( f \) is continuous and injective on \( D \) then \( f \) must be strictly monotone.

This is true. It is theorem 3.14 in chapter 3.

b) If \( f \) is continuous on \( \text{dom} f \) and \( \text{dom} f \) is bounded, then \( \{f(x_n)\} \) must be bounded for any \( \{x_n\} \subset \text{dom} f \).

This is false. If \( f(x) = \frac{1}{x} \) is continuous on the bounded domain \( (0, 1) \) and \( x_n = \frac{1}{n} \) is included in \( (0, 1) \) but \( f(x_n) = n \) is not bounded.

4. Suppose \( f \) is continuous on \([0, 2]\) and \( 0 \leq f(x) \leq 2 \) for all \( x \), \( 0 \leq x \leq 2 \). Show that \( f(x) = x \)

for some \( x, 0 \leq x \leq 2 \).

Let \( g(x) = x - f(x) \)  Then \( g \) is continuous on \([0, 2]\) and since \( 0 \leq f(x) \leq 2 \), it follows that \( g(0) = -f(0) \leq 0 \), while \( g(2) = 2 - f(2) \geq 0 \). Then by the intermediate value theorem, we know that for some \( x \) in \([0, 2]\) \( g(x) = 0 \), which is to say, \( f(x) = x \).

5. Tell which of the following functions are continuous and which are uniformly continuous on the domains indicated

(a) \( \frac{1}{x(1-x)} \) is continuous on \((0, 1)\) by arithmetic with continuous functions theorem but the limits at \( x = 0 \) and \( x = 1 \) do not exist so it is not uniformly continuous.
(b) \( \frac{x}{1+x^2} \) is continuous on \((0, \infty)\) by arithmetic with continuous functions theorem and the limits at \(x = 0\) and \(x = \infty\) both exist so it is also uniformly continuous.

6. Suppose \(f\) is continuous at \(x_0 \in \text{dom } f\) and \(f(x_0) = y_0\). Then explain why every point of the set \(\{x : y_0 - 1 < f(x) < y_0 + 1\}\) is an interior point.

This is the topological defn of continuity. For \(\varepsilon = 1\) there is a \(\delta > 0\) such that \(f(x) \in (y_0 - 1, y_0 + 1)\) when \(x \in N_\delta(x_0)\).

That is to say \(\{x : y_0 - 1 < f(x) < y_0 + 1\} = N_\delta(x_0) = (x_0 - \delta, x_0 + \delta)\) is open.

7. Let \(S = \{x : 0 \leq x^2 \leq 4\}\) and suppose \(\{x_n\} \subset S\). Show that \(\{x_n\}\) contains a subsequence converging to a point of \(S\). Be sure to justify all your statements.

\(S = \{x : 0 \leq x^2 \leq 4\} = \{0 \leq x \leq 2\}\) is a closed bounded set. If \(\{x_n\} \subset S\) then \(\{x_n\}\) is bounded and contains a convergent subsequence by the B-W theorem. Since \(S\) is closed, the limit point belongs to \(S\).

8. Suppose \(f\) is strictly monotone on \(\text{dom } f\). Let \(x, y \in \text{dom } f\). If \(x \neq y\) then \(x < y\) or \(y < x\) and it follows that \(f(x) < f(y)\) or else \(f(x) > f(y)\). In any case, \(f(x) \neq f(y)\). What is this a proof of?

This is a proof of the fact that if \(f\) is strictly monotone then \(f\) is injective.