6) Let \( F \) be a field and \( f, g \in F[x, y] \). Suppose that \((x_0, y_0)\) is a common solution to \( f(x, y) = 0 \), \( g(x, y) = 0 \).
   a) Show that \( y_0 \) must be a root of \( \text{res}_y(f(x, y), g(x, y)) \).
   b) Describe a method, based on a), that solves a system of polynomial equations by first eliminating all variables but one using resultants, then solves this polynomial in one variable, and finally uses back-substitution to find all solutions.
   c) Use the method of b) to find all rational solution to the following system of equations:
      \[
      \{x^2y - 3xy^2 + x^2 - 3xy = 0, x^3y + x^3 - 4y^2 - 3y + 1 = 0\}
      \]

The GAP functions \texttt{Resultant} and \texttt{Factors} might be helpful. See the online help for details.

7) Let \( F \) be a field and \( f(x), g(x) \in F[x] \). Suppose that \( \alpha, \beta \in \bar{F} \) (the algebraic closure) such that \( f(\alpha) = 0, g(\beta) = 0 \).
   a) Show that \( \text{res}_y(f(x - y), g(y)) \) has a root \( \alpha + \beta \). (Note: Similar expressions exist for \( \alpha - \beta, \alpha \cdot \beta \) and \( \alpha/\beta \).)
   b) Construct a polynomial \( f \in \mathbb{Q}[x] \) such that \( \mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}(\sqrt{3}, \sqrt{2}) \).

8) Let \( F \) be a perfect field, \( f(x) \in F[x] \) irreducible, \( K \supseteq F \) the splitting field of \( f \) over \( F \) and \( \theta \in K \) a root of \( f \). We extend the elements of \( \text{Gal}(K/F) \) to \( K[x] \) by acting on the coefficients of a polynomial. (So for example if \( \sigma \) is complex conjugation then \( \sigma(x^3 + ix + 2) = x^3 - ix + 2 \).) For \( g \in K[x] \) define the Norm of \( g \) as:
   \[
   N(g) = \prod_{\sigma \in \text{Gal}(K/F)} \sigma(g)
   \]
   a) Show that \( N(g) \in F[x] \).
   b) For \( g \in F(\theta)[x] \) define a polynomial \( \tilde{g} \in F[x, y] \) by substituting \( y \) for \( \theta \).
      Show that \( N(g) = \text{res}_x(f(y), \tilde{g}(x, y)) \).
   c) Suppose that \( g \) is given as in b) and that \( \gamma \) is a root of \( g \) in the algebraic closure. Show that \( \gamma \) is a root of \( N(g) \).
   d) Let \( \alpha \) be a root of \( x^3 + 2 \) over \( \mathbb{Q} \) and \( \beta \) a root of \( x^2 + \alpha \cdot x + \alpha^2 \) over \( \mathbb{Q}(\alpha) \). Compute a rational polynomial with root \( \alpha \).
      (Note: This approach permits to reduce iterated algebraic extensions to simple extensions.)

9) The discriminant of a polynomial \( f \in F[x] \) of degree \( m \) with leading coefficient \( a \) is defined as
   \[
   \text{disc}(f) = (-1)^{\frac{m(m-1)}{2}} \text{res}(f(x), f'(x))/a
   \]
   where \( f'(x) \) denotes the derivative as defined in Analysis.
   a) Show that \( \text{disc}(f) = 0 \) if and only if \( f \) has multiple roots (i.e. a factor \((x - \alpha)^2 \) over the complex numbers. (Hint: Proposition 13.33 in Dummit&Foote.)
   b) Let \( f(x) \in \mathbb{Z}[x] \) be irreducible. Show that there are only finitely many primes \( p \), such that the reduction of \( f \) modulo \( p \) has multiple roots.
   c) Determine all such primes for the polynomial \( x^7 + 15x^6 + 12 \).