22) Let $G = D_8$ the group of symmetries of the square and $H = Q_8$ the quaternion group defined in problem 8. Both groups have order 8.
   a) Show (briefly) that both $G$ and $H$ have order 8, 5 conjugacy classes, and an abelian factor group isomorphic to $C_2 \times C_2$.
   b) Determine the irreducible characters of $C_2 \times C_2$.
   c) Show that both $G$ and $H$ have 4 irreducible characters of degree 1 and one character of degree 2. Using the result of b), determine the irreducible characters of degree 1.
   d) Show that $G$ and $H$ have (up to permuting characters or classes) have the same character table.

23) Let $G$ be a finite group and $e$ be the exponent of $G$, that is the least common multiple of all element orders in $G$. Let $\epsilon$ be a primitive $e$-th root of unity.
   a) Show that for any character $\chi$ of $G$ and any $g \in G$ we have that $\chi(g) \in \mathbb{Q}(\epsilon)$.
   b) Show that for any character $\chi$ of $G$ and any $g \in G$ we have that $\chi(g^{-1}) = \overline{\chi(g)}$, with $\overline{\cdot}$ denoting complex conjugation.
   (Hint: A theorem from linear algebra shows that every matrix can be upper-triangulized, if the characteristic polynomial splits. What does this say about the eigenvalues of $g$ and $g^{-1}$?)

24) Let $V, W$ be $K$ vector spaces of dimensions $m$, respectively $n$, with bases $\{v_i\}_{i=1}^m$, respectively $\{w_j\}_{j=1}^n$. We consider the tensor product $V \otimes W$ as an $m \cdot n$-dimensional vector space with basis vectors denoted as $v_i \otimes w_j$ for $1 \leq i \leq m, 1 \leq j \leq n$ and define the linear map $\otimes: V \times W \rightarrow V \otimes W$ by

$$\otimes:\left(\sum_i \lambda_i v_i, \sum_j \mu_j w_j\right) \mapsto \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j).$$

(This construction has been introduced in 567.)

Now suppose that for a group $G$ we have that $V, W$ are $KG$-modules. We define a map $(V \otimes W) \times G \rightarrow V \otimes W$ by setting

$$\left(\sum_{i,j} \lambda_{i,j} v_i \otimes w_j\right) g := \sum_{i,j} \lambda_{i,j} (v_i g \otimes w_j g).$$

(Note that the $\otimes$ on the left hand side is a symbol for basis vectors, while it is the linear map $V \times W \rightarrow V \otimes W$ on the right hand side.)
   a) Show that for arbitrary $v \in V, w \in W$ and $g \in G$ we have that

$$(v \otimes w)g = vw g \otimes w g$$

b) Show that with this definition $V \otimes W$ becomes a $KG$ module, that is $G$ acts on $V \otimes W$.
   c) Give an example, showing that the property in a) does not hold if $g$ is replaced by arbitrary elements of $KG$. 