15) Let $G$ be a finite group and $A = \mathbb{C}G$. Find a basis for $\text{Hom}_A(A_A, A_A)$.

18) Let $G = \text{SL}_3(2)$. We want to determine the degrees of the irreducible $\mathbb{C}$-representations of $G$.

a) Determine the irreducible polynomials of degree up to 3 over $\mathbb{F}_2$. Based on this show that there are 6 classes of Elements in $G$. What are their element orders?

b) Show that $G = G'$. (Thus there is only one linear representation.)

c) Determine the possibilities for writing $|G| - 1$ as a sum of 5 squares of divisors of $|G|$. You might find the following GAP command helpful, where size = $|G| - 1$ and list is the divisors of $|G|$

\[
\text{Filtered(UnorderedTuples(list,5),x->Sum(x,y->y^2)=size)};
\]

d) Determine the degrees of the irreducible representations of $G$, assuming the existence of an irreducible representation of degree 6, which we will construct later.

(The irreducible representation of degree 6 comes from the doubly-transitive action of $G$ on the nonzero vectors of $\mathbb{F}_2^5$.)

20) (If you have not seen the ring of algebraic integers – we shall only require the result from c) later on) Let $R$ be an integral domain with quotient field $F$. Let $\bar{F}$ be the algebraic closure of $F$. We then can write polynomials over $R$ as product of linear factors over $\bar{F}$.

Let $f, g \in R[x]$ such that $f(x) = a \cdot \prod_{i=1}^{m}(x - \alpha_i)$ and $g(x) = b \cdot \prod_{j=1}^{n}(x - \beta_j)$ in $\bar{F}[x]$. We define the Resultant (with respect to $x$) of $f$ and $g$ as

\[
\text{Res}_x(f, g) = a^n b^m \prod_{i,j} (\alpha_i - \beta_j) = (-1)^{mn} b^m \prod_{j} f(\beta_j) = a^n \prod_{i} g(\alpha_i)
\]

(you may assume equality of these 3 expressions without proof. You can find a proof in Dummit, Foote: Abstract Algebra, p.619ff, exercise 29ff.)

a) Show that $\text{Res}_x(f, g) \in R$. (Hint: Show that $\text{Res}$ is a symmetric $R$-polynomial in the $\alpha_i$, thus by the fundamental theorem of elementary symmetric functions it can be written as an $R$-polynomial in the in the coefficients of $f$.)

b) We consider $R[x] \subset R[x][y]$. (Thus, having $R[x]$ as the base ring, $\text{Res}_y(f(x, y), g(x, y) \in R[x]$.)

Show that, if $f, g \in R[x]$ with $f(\alpha) = 0$ and $g(\beta) = 0$ that

i) $\alpha + \beta$ is a root of $\text{Res}_y(f(x - y), g(y))$.

ii) $\alpha \cdot \beta$ is a root of $\text{Res}_y(y^m f(x/y), g(y))$. (Note that $y^m f(x/y) \in R[x, y]$.)

c) Now consider $R = \mathbb{Z}$. An element $\alpha \in \mathbb{C}$ is called an algebraic integer if there exists a monic (i.e. leading coefficient is 1) polynomial $f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x]$, such that $f(\alpha) = 0$.

Show that the set of algebraic integers is closed under addition, subtraction and multiplication, i.e. that it forms a ring.