10) (GAP) a) Let $G = A_5$ and $F = \mathbb{F}_9 = GF(9)$. Determine with the MeatAxe (but without using the MTX.CollectedFactors command) the irreducible $FG$-modules of dimension 3 up to isomorphism.
   b) Also determine (you may use the MTX.CollectedFactors command here) the irreducible $F_iG$ modules of dimension 3 up to isomorphism. Compare with the result of a).

11) Let $F = \mathbb{F}_2$, $m = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{F}_2^{4 \times 4}$ and $G = \{m\}$. Then $|m| = 7$ (you do not need to show this).

Let $V = \mathbb{F}_3^3$ be the $FG$ module obtained by the matrix action (called the natural module) and $W = \langle (0,0,0,1) \rangle \leq V$ be a 1-dimensional $FG$-submodule. (You do not need to show any of the statements so far.)

Using the method from the proof of Maschke’s theorem, construct a $FG$-submodule $S$ such that $V = W \oplus S$.

12) (Examples to show the necessity of the conditions for Maschke’s theorem)

   a) Let $G = \langle g_1, \ldots, g_k \rangle$ a finite group, $F$ a field and $\varphi, \psi : G \to GL_n(F)$ be two equivalent irreducible representations of a finite group $G$ (i.e. there exists a matrix $M \in GL_n(F)$ such that for all $g \in G$:
   $M^{-1}(g^\varphi)M = g^\psi$.
   
   The aim of this problem is to show a method to test constructively for such an isomorphism.
   a) Let $\omega \in FG$ be an element such that the nullity of $\omega^\varphi$ (i.e. the dimension of $ker \omega^\varphi$) is 1. Show that the nullity of $\omega^\psi$ is also 1.
   b) Let $x \in ker(\omega^\varphi)$ and $y \in ker(\omega^\psi)$. Show that there exists $\alpha \in F$ such that $x \cdot M = \alpha \cdot y$.
   c) We now start a spinning algorithm for $(g_1^\varphi, \ldots, g_k^\varphi)$, starting with $x_i$ obtaining (as $\varphi$ is irreducible) a basis $B = \{x_1 := x, x_2, \ldots, x_n\}$ of $F^n$.

   We also start (in parallel) a spinning algorithm for $(g_1^\psi, \ldots, g_k^\psi)$, starting with $y$ and obtain a second basis $C = \{y_1 := y, y_2, \ldots, y_n\}$ of $F^n$.

   Show that:
   
   1. $x \cdot M = \alpha \cdot y$, where $\alpha$ is as defined in b).
   2. For all $i, j, l$ we have that $x_i \cdot g_j^\varphi \in \text{Span}(x_1, \ldots, x_n) \iff y_i \cdot g_j^\psi \in \text{Span}(y_1, \ldots, y_n)$

   (Hint: It might be easier to prove this by using GAP functions in one induction. The second statement will be needed to show that both spinning algorithms will define new elements $x_i$ at exactly the same time in the “same” way, which then proves the first statement.)
   d) Let $N = \varphi[\text{id}]_C$ the matrix for base change from (we act on the right) $B$ to $C$. Show that $N^{-1} a^\varphi N = a^\psi$, i.e. that $N$ can serve as base change showing the equivalence of $\varphi$ and $\psi$.

Remark: If the representations are not equivalent, either $\omega^\varphi$ and $\omega^\psi$ will have different nullity, or the matrix $\varphi[\text{id}]_C$ will not map $a^\varphi$ to $a^\psi$. This method thus gives a constructive isomorphism test for irreducible modules. The command MTX.Isomorphism in GAP does exactly this test.