Let $F$ be a field and $x_1, \ldots, x_n$ be indeterminates. A polynomial $f \in F[x_1, \ldots, x_n]$ is called symmetric, if for any permutation $\sigma$ of the indeterminates we have that $\sigma(f) = f$.

Obviously, the following functions (called elementary symmetric polynomials) are symmetric:

$$
S_1(x_1, \ldots, x_n) = x_1 + x_2 + x_3 + \cdots + x_n \\
S_2(x_1, \ldots, x_n) = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\
S_3(x_1, \ldots, x_n) = \sum_{i<j<k} x_1x_jx_k \\
\vdots \\
S_n(x_1, \ldots, x_n) = x_1x_2 \cdots x_n
$$

The aim of this exercise is to prove the fundamental theorem of elementary symmetric polynomials:

**Theorem:** Let $f \in F[x_1, \ldots, x_n]$ symmetric, then exists $g \in F[x_1, \ldots, x_n]$ such that

$$
f(x_1, \ldots, x_n) = g(S_1(x_1, \ldots, x_n), S_2(x_1, \ldots, x_n), \ldots, S_n(x_1, \ldots, x_n)).
$$

**Example:** $x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1x_2x_3^3 + 2x_1x_2^2 + 2x_1x_3 + x_2^3 + 2x_2x_3 + x_3^2 = S_2^2 + S_2 * S_3$.

We define an ordering on monomials by setting: $x_1^{d_1}x_2^{d_2} \cdots x_n^{d_n} > y_1^{e_1}y_2^{e_2} \cdots y_n^{e_n}$ if and only if for some $i$: $d_j = e_j$ for $j < i$ and $d_i > e_i$ (lexicographic comparison of the exponent vectors).

a) Suppose that $d_1 \geq d_2 \geq \cdots \geq d_n$ is a sequence of integers. Show that $x_1^{d_1}x_2^{d_2} \cdots x_n^{d_n}$ is the largest (with respect to this ordering) term of the polynomial

$$
T_{d_1, \ldots, d_n}(x_1, \ldots, x_n) := S_1^{d_1-d_2-d_3-\cdots-d_n} \cdots S_1^{d_{n-2}-d_{n-1}-d_n} \cdot S_2^{d_{n-2}-d_{n-3} \cdots -d_n} \cdot S_3^{d_{n-3}-d_{n-4} \cdots -d_n} \cdot \cdots \cdot S_n
$$

b) Suppose that $f(x_1, \ldots, x_n)$ is symmetric and $c \cdot x_1^{d_1}x_2^{d_2} \cdots x_n^{d_n}$ is the largest (with respect to this ordering) term of $f$ and let $T_{d_1, \ldots, d_n}(x_1, \ldots, x_n)$ for these $d_i$ as in a). Show that the largest monomial of $f(x_1, \ldots, x_n) - c \cdot T_{d_1, \ldots, d_n}(x_1, \ldots, x_n)$ is strictly smaller than $x_1^{d_1}x_2^{d_2} \cdots x_n^{d_n}$.

c) Show that by iterating the process in b) you can build the polynomial $g$ as claimed in the theorem.

d) Express

$$
f = x^2y + x^2z + 3xyz + y^2x + y^2z + z^2x + yz^2 - 2x^3y^3z - 4x^3y^2z^2 - 4x^2y^3z^2 - 2x^3y^3z^3 - 2xy^3z^3
$$

as a polynomial in the elementary symmetric polynomials in $x$, $y$ and $z$. 