29) Show that two $3 \times 3$ matrices are similar if and only if they have the same characteristic polynomial and the same minimal polynomial. Give a counterexample to this assertion for $4 \times 4$ matrices.

30) Determine the characteristic and the minimal polynomial of the following matrix over $\mathbb{F}_2$:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

31) Let

$$
A := \begin{pmatrix}
-3 & -5 & 6 \\
-16 & -19 & 24 \\
-16 & -20 & 25
\end{pmatrix}, \quad B := \begin{pmatrix}
1 & 1 & -1 \\
0 & 2 & -1 \\
0 & 1 & 0
\end{pmatrix}
$$

a) Determine the rational normal form of $A$ and find an invertible matrix $Q \in \mathbb{Q}^{3 \times 3}$ such that $Q^{-1}AQ$ is in rational normal form.
b) Determine the characteristic polynomial and the minimal polynomial of $A$.
c) Show that $A$ and $B$ are similar.
d) Find an invertible matrix $P \in \mathbb{Q}^{3 \times 3}$ such that $P^{-1}AP = B$.

32) Show that $x^6 - 1 = (x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1)$ over $\mathbb{F}_5$. Using this determine (representatives of) the conjugacy classes of $\text{GL}_4(5)$ of elements of order 2, 3, or 6.

33) Let $F$ be a field and let $A \in F^{n \times n}$. Show that $A \sim A^T$.

34) Compute the Jordan Canonical Form of

$$
A := \begin{pmatrix}
1 & 3 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 18 & -5 & 0 & 6 & -25 & 0 & -2 \\
0 & 64 & -18 & 0 & 24 & -100 & 0 & -8 \\
-2 & 6 & 0 & 2 & 0 & -1 & -2 & 0 \\
0 & 116 & -34 & 0 & 43 & -170 & 0 & -14 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 45 & -13 & 0 & 15 & -65 & 1 & -5 \\
0 & 322 & -94 & 0 & 114 & -470 & 0 & -37
\end{pmatrix}
$$

You may use a computer algebra system for calculating matrix powers and nullspaces.
Computing the Jordan Canonical Form

Given a matrix $A \in F^{n \times n}$ whose characteristic polynomial splits into linear factors we want to determine its Jordan Canonical form as well as a transforming matrix. We let $K^i = \ker(A - \lambda \cdot I)^i$ and observe that the basis vectors for a basis in JCF that are at position $i$ in a block are in $K^i$ but not in $K^{i-1}$. We call $K^n$ the generalized eigenspace for $\lambda$.

1. Determine the Eigenvalues $\lambda_k$ of the matrix $A$ (for example as roots of the characteristic polynomial). For each eigenvalue $\lambda$ perform the following calculation (which gives a basis of the generalized eigenspace of $\lambda$, the whole basis will be obtained by concatenating the bases obtained for the different $\lambda_k$).

   Again we write $K^i = \ker(A - \lambda \cdot I)^i$.

2. Calculate $e_i = \dim K^i$ until the sequence becomes stationary. (The largest $e_i$ is the dimension of the generalized eigenspace.)

3. Let $f_i = e_i - e_{i-1}$. Then $e_i$ gives the number of Jordan blocks that have size at least $i$. (As long as we only want to know the Jordan form, we thus could stop here.)

   We now build a basis in sequence of descending $i$. Let $B = []$ and $i = \max\{i \mid f_i > 0\}$.

4. (Continue growing the existing Jordan blocks) For each vector list $(s_1, \ldots, s_m)$ in $B$, append the image $(A - \lambda \cdot I) \cdot s_m$ of its last element to the list.

5. (Start new Jordan block of size $i$) If $f_i - f_{i-1} = m > 0$ (then the images of the vectors obtained so far do not span $K^i$) let $W = \text{Span}(K^{i-1}, s_1, \ldots, s_k)$ where the $s_j$ run through the elements in all the lists obtained so far. Extend a basis of $W$ to a basis of $K^i$ by adding $m$ linearly independent vectors $b_1, \ldots, b_m$ in $K^i - K^{i-1}$ to it.

   The probability is high (why?) that any $m$ linear independent basis vectors of $K^i$ fulfill this property. To verify it, choose a basis for $K^{i-1}$, append the $s_j$ and then append the $b_i$. Then show that the resulting list is linearly independent.

   (The generic method would be to extend a basis of $W$ to a basis of $K^i$ and take the vectors by which the basis got extended.)

6. For each such vector $b_i$ add a list $[b_i]$ to $B$.

7. If the number of vectors in the lists in $B$ is smaller than the maximal $e_i$, then decrement $i$ and go to step (4).

8. Concatenate the reverses of the lists in $B$. This is the part of the basis corresponding to eigenvalue $\lambda$. 
For example, let

\[
A := \begin{pmatrix}
59 & -224 & 511 & -214 & 4 \\
16 & -61 & 139 & -58 & 1 \\
6 & -24 & 13 & -20 & 0 \\
13 & -52 & 110 & -43 & 0 \\
13 & -52 & 110 & -43 & 0
\end{pmatrix}.
\]

Its characteristic polynomial is \((x - 1)^4\). We get the following nullspace dimensions and their differences:

<table>
<thead>
<tr>
<th>(i)</th>
<th>(e_i = \dim K^i)</th>
<th>(f_i = e_{i+1} - e_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

At this point we know already the shape of the Jordan Canonical form of \(A\) (2 blocks of size 1 or larger, 2 blocks of size 2 or larger, 1 block of size 3 or larger. I.e. One block of size 2 and one block of size 3), but let us compute the explicit base change:

We start at \(i = 3\) and set \(B = \emptyset\). We have that \(K^3 = \mathbb{R}^5\) and

\[
K^2 = \text{Span}(4, 1, 0, 0, 0)^T, (-15/2, 0, 1, 0, 0)^T, (3, 0, 0, 1, 0)^T, (0, 0, 0, 0, 1)^T,
\]

As \(f_3 = 1\) we need to find only one basis vector and pick \(b_1 := (1, 0, 0, 0, 0)^T\) as first basis vector (an almost random choice, we only have to make sure it is not contained in \(K^2\), which is easy to verify) and add the list \([b_1]\) to \(B\).

In step \(i = 2\) we first compute the image \(b_2 := (A - 1)b_1 = (58, 16, 6, 13, -4)^T\) and add it to the list.

Furthermore, as \(f_2 > f_3\), we have to get another basis vector in \(K^2\), but not in the span of \(K^3\) and \(b_2\). We pick \(b_3 = (4, 1, 0, 0, 0)^T\) from the spanning set of \(K^2\), and verify that it indeed fulfills the conditions. We thus have \(B = [b_1, b_2, [b_3]]\).

In step \(i = 1\) we now compute images again \(b_4 := (A - 1)b_2 = (48, 12, 4, 10, 0)^T\) and (from the second list) \(b_5 := (A - 1)b_3 = (8, 2, 0, 0, -4)^T\).

As \(f_1 = f_2\) no new vectors are added.

As a result we get \(B = [b_1, b_2, b_4, b_5, b_3]\).

Finally we concatenate the reversed basis vector lists and get the new basis \([b_4, b_2, b_1, b_5, b_3]\).

We thus have the base change matrix

\[
S := \begin{pmatrix}
48 & 58 & 1 & 8 & 4 \\
12 & 16 & 0 & 2 & 1 \\
4 & 6 & 0 & 0 & 0 \\
10 & 13 & 0 & 0 & 0 \\
0 & -4 & 0 & -4 & 0
\end{pmatrix}.
\]

It is easily verified that \(S^{-1}AS = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}\).